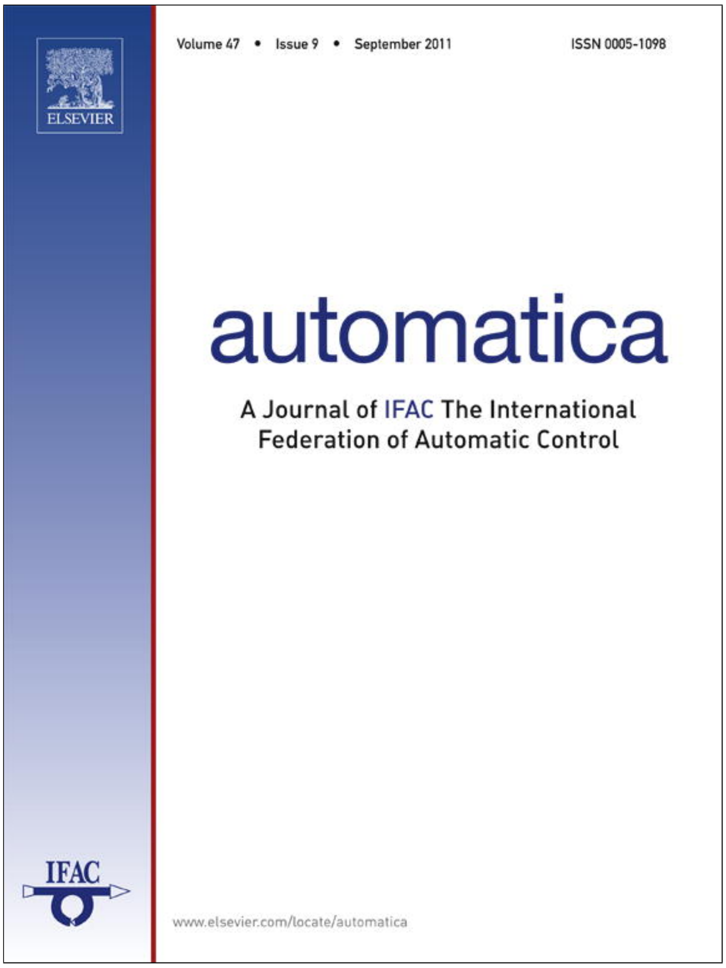
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Brief paper

Generalised theory on asymptotic stability and boundedness of stochastic functional differential equations☆

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ABSTRACT

Asymptotic stability and boundedness have been two of most popular topics in the study of stochastic functional differential equations (SFDEs) (see e.g. Appleby and Reynolds (2008), Appleby and Rodkina (2009), Basin and Rodkina (2008), Khasminskii (1980), Mao (1995), Mao (1997), Mao (2007), Rodkina and Basin (2007), Shu, Lam, and Xu (2009), Yang, Gao, Lam, and Shi (2009), Yuan and Lygeros (2005) and Yuan and Lygeros (2006)). In general, the existing results on asymptotic stability and boundedness of SFDEs require (i) the coefficients of the SFDEs obey the local Lipschitz condition and the linear growth condition; (ii) the diffusion operator of the SFDEs acting on a $C^{2,1}$-function be bounded by a polynomial with the same order as the $C^{2,1}$-function. However, there are many SFDEs which do not obey the linear growth condition. Moreover, for such highly nonlinear SFDEs, the diffusion operator acting on a $C^{2,1}$-function is generally bounded by a polynomial with a higher order than the $C^{2,1}$-function. Hence the existing criteria on stability and boundedness for SFDEs are not applicable and we see the necessity to develop new criteria. Our main aim in this paper is to establish new criteria where the linear growth condition is no longer needed while the up-bound for the diffusion operator may take a much more general form.

1. Introduction

Systems in many branches of science and industry do not only depend on the present state but also the past ones. Stochastic functional differential equations (SFDEs) have been widely used to model such systems (see e.g. Kolmanovskii and Myshkis (1992), Kolmanovskii and Nosov (1981), Mao (1991), Mao (1994), Mao (2007), Mao and Yuan (2006) and Mohammed (1986)). Asymptotic stability and boundedness have been two of most popular topics in the study of SFDEs and there is an extensive literature in these areas (see e.g. Appleby and Reynolds (2008), Appleby and Rodkina (2009), Basin and Rodkina (2008), Khasminskii (1980), Mao (1995) Mao (1997), Mao (2007), Rodkina and Basin (2007), Shu et al. (2009), Yang et al. (2009), Yuan and Lygeros (2005) and Yuan and Lygeros (2006)).

In general, an SFDE has the form

$$dx(t) = f(x_t, t)dt + g(x_t, t)dB(t) \tag{1}$$

on $t \geq 0$ with initial data $x_0 = \xi \in C([-\tau, 0]; R^n)$, where $f : C([-\tau, 0]; R^n) \times R_+ \to R^n$ and $g : C([-\tau, 0]; R^n) \times R_+ \to R^{n\times m}$. (The notations used in this section will be explained in Section 2). In general, both f and g are required to obey the standard local Lipschitz condition and the linear growth condition (see e.g. Kolmanovskii and Myshkis (1992), Mao (2007) and Mohammed (1986)). One of the useful stability criteria is the following result:

Theorem 1 (*Mao and Yuan (2006, Theorem 8.7 on page 308)*). *Assume that both f and g satisfy the local Lipschitz condition and the linear growth condition. Let $V \in C^{2,1}(R^n \times R_+; R_+)$, $w \in \mathcal{W}([-\tau, 0]; R_+)$ and $p, \lambda_1, \lambda_2, c_1, c_2$ be all positive constants with $\lambda_1 > \lambda_2$. If*

$$c_1|x|^p \leq V(x, t) \leq c_2|x|^p, \quad \forall (x, t) \in R^n \times R_+,$$

and

$$\mathcal{L}V(\varphi, t) \leq -\lambda_1|\varphi(0)|^p + \lambda_2 \int_{-\tau}^{0} w(u)|\varphi(u)|^p du$$

for all $(\varphi, t) \in C([-\tau, 0]; R^n) \times R_+$, then for every initial data $\xi \in C([-\tau, 0]; R^n)$, the solution of Eq. (1) obeys

$$\lim_{t\to\infty} \frac{1}{t} \log(E|x(t)|^p) \leq -\lambda,$$

where $\lambda > 0$ is the unique root to $\lambda_2 e^{\lambda\tau} = \lambda_1 - \lambda_2$.

However, there are many SFDEs to which the above theorem and other existing stability criteria cannot be applied. For example,

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consider the scalar SFDE

$$\mathrm{d}x(t) = [-x(t) - x^3(t) + (D(x_t))^2]\mathrm{d}t + \bar{D}(x_t)\mathrm{d}B(t) \tag{2}$$

with initial data $x_0 = \xi \in C([-\tau, 0]; R)$, where $B(t)$ is a scalar Brownian motion and both D and $\bar{D}$ are bounded linear operators from $C([-\tau, 0]; R)$ to R. By the classical Riesz theorem (see e.g. Natanson (1964)), there is a function of bounded variation $A(\cdot)$ on $[-\tau, 0]$ with $A(-\tau) = 0$ such that

$$D(\varphi) = \int_{-\tau}^{0} \varphi(\theta)\mathrm{d}A(\theta), \quad \varphi \in C([-\tau, 0]; R).$$

It is well-known that any function of bounded variation can be written as the difference of two non-decreasing functions. In other words, we can write $A = A_1 - A_2$, where both A_1 and A_2 are non-decreasing functions on $[-\tau, 0]$ with $A_1(-\tau) = A_2(-\tau) = 0$. Hence, for $\varphi \in C([-\tau, 0]; R)$,

$$D(\varphi) = \int_{-\tau}^{0} \varphi(\theta)\mathrm{d}A_1(\theta) - \int_{-\tau}^{0} \varphi(\theta)\mathrm{d}A_2(\theta),$$

which implies,

$$|D(\varphi)| \le K \int_{-\tau}^{0} |\varphi(\theta)|\mathrm{d}\mu(\theta) \tag{3}$$

where $K = A_1(0) + A_2(0)$ and $\mu = (A_1 + A_2)/K$ which is a probability measure on $[-\tau, 0]$. Similarly, there is a probability measure $\bar{\mu}$ on $[-\tau, 0]$ and a positive constant $\bar{K}$ such that

$$|\bar{D}(\varphi)| \le \bar{K} \int_{-\tau}^{0} |\varphi(\theta)|\mathrm{d}\bar{\mu}(\theta). \tag{4}$$

For illustration, we assume that $K = \bar{K} = 1$. When we attempt to apply the existing theory of SFDEs, we encounter two problems:

(i) The drift coefficient $f(\varphi, t) := -\varphi(0) - \varphi^3(0) + (D(\varphi))^2$ does not obey the linear growth condition although they are locally Lipschitz continuous. To the authors' best knowledge, there is so far no result that shows that this equation has a unique global solution for the given initial data.

(ii) Even if there is no problem with the existence of the global solution, we still encounter another problem when we attempt to apply e.g. Theorem 1 to deduce the exponential decay of the solution. To see this new problem, let us set $V(x, t) = x^2$. Then

$$\begin{aligned}\mathcal{L}V(\varphi, t) &= 2\varphi(0)[-\varphi(0) - \varphi^3(0) + (D(\varphi))^2] + (\bar{D}(\varphi))^2 \\ &\le -\frac{7}{5}|\varphi(0)|^2 - 2|\varphi(0)|^4 + \frac{5}{3}\int_{-\tau}^{0} |\varphi(\theta)|^4\mathrm{d}\mu(\theta) \\ &\quad + \int_{-\tau}^{0} |\varphi(\theta)|^2\mathrm{d}\bar{\mu}(\theta).\end{aligned} \tag{5}$$

The terms $-2|\varphi(0)|^4 + \frac{5}{3}\int_{-\tau}^{0} |\varphi(\theta)|^4\mathrm{d}\mu(\theta)$ with a higher order than the order of $V(x, t) = x^2$ appear on the right-hand side and these prevent Theorem 1 from being used.

It is due to these problems that we see the necessity to develop new stability criteria for the SFDE (1) where the linear growth condition may not hold while the bound on the operator $\mathcal{L}V$ may take a much more general form.

2. Existence-and-uniqueness theorem

Throughout this paper, unless otherwise specified, we use the following notation. Let $|\cdot|$ be the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^TA)}$. Let $R_+ = [0, \infty)$ and $\tau > 0$. Denote by $C([-\tau, 0]; R^n)$ the family of continuous functions φ from $[-\tau, 0]$ to R^n with the norm $\|\varphi\| = \sup_{-\tau\le\theta\le 0} |\varphi(\theta)|$. Denote by $\mathcal{W}([-\tau, 0]; R_+)$ the family of mappings $w : [-\tau, 0] \to R_+$ such that $\int_{-\tau}^{0} w(u)\mathrm{d}u = 1$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $B(t) = (B_1(t), \ldots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. If $x(t)$ is an R^n-valued stochastic process on $t \in [-\tau, \infty)$, we let $x_t = \{x(t + \theta) : -\tau \le \theta \le 0\}$ for $t \ge 0$ whence x_t is a $C([-\tau, 0]; R^n)$-valued stochastic process.

Consider a nonlinear SFDE

$$\mathrm{d}x(t) = f(x_t, t)\mathrm{d}t + g(x_t, t)\mathrm{d}B(t), \quad t \ge 0, \tag{6}$$

where $f : C([-\tau, 0]; R^n) \times R_+ \to R^n$ and $g : C([-\tau, 0]; R^n) \times R_+ \to R^{n\times m}$. In order to solve the equation we need to know the initial data and we assume that they are given by

$$x_0 = \xi \in C([-\tau, 0]; R^n). \tag{7}$$

The well-known conditions imposed for the existence and uniqueness of the global solution are the Local Lipschitz condition and the linear growth condition (see e.g. Mao (1991), Mao (1994), Mao (2007) and Mohammed (1986)). To be precise, let us state these conditions.

Assumption 1 (*The Local Lipschitz Condition*)**.** For each integer $i \ge 1$ there is a positive constant K_i such that

$$|f(\varphi, t) - f(\phi, t)|^2 \vee |g(\varphi, t) - g(\phi, t)|^2 \le K_i\|\varphi - \phi\|^2$$

for those $\varphi, \phi \in C([-\tau, 0]; R^n)$ with $\|\varphi\|\vee\|\phi\| \le i$ and any $t \in R_+$.

Assumption 2 (*The Linear Growth Condition*)**.** There is a positive constant $\bar{K}$ such that

$$|f(\varphi, t)|^2 \vee |g(\varphi, t)|^2 \le \bar{K}(1 + \|\varphi\|^2)$$

for all $(\varphi, t) \in C([-\tau, 0]; R^n) \times R_+$.

In this paper we shall retain the local Lipschitz condition but replace the linear growth condition by a more general condition in order to guarantee the existence of a unique global solution. To state this general condition, we need a few more notations. Denote by $C(R^n \times [-\tau, \infty]; R_+)$ the family of continuous functions from $R^n \times [-\tau, \infty)$ to R_+. Let $C^{2,1}(R^n \times [-\tau, \infty); R_+)$ denote the family of all continuous non-negative functions $V(x, t)$ defined on $R^n \times [-\tau, \infty)$ such that they are continuously twice differentiable in x and once in t. Given $V \in C^{2,1}(R^n \times [-\tau, \infty); R_+)$, we define the functional $\mathcal{L}V : C([-\tau, 0]; R^n) \times R_+ \to R$ by

$$\begin{aligned}\mathcal{L}V(\varphi, t) &= V_t(\varphi(0), t) + V_x(\varphi(0), t)f(\varphi, t) \\ &\quad + \frac{1}{2}\mathrm{trace}[g^T(\varphi, t)V_{xx}(\varphi(0), t)g(\varphi, t)],\end{aligned}$$

where $V_x(x, t) = (V_{x_1}(x, t), \ldots, V_{x_n}(x, t))$ and $V_{xx}(x, t) = (V_{x_ix_j}(x, t))_{n\times n}$. Let us emphasise that $\mathcal{L}V$ is defined on $C([-\tau, 0]; R^n) \times R_+$ while V on $R^n \times [-\tau, \infty)$.

Motivated by the example discussed in Section 1, i.e. the SFDE (2), let us now propose our more general conditions.

Assumption 3. There are two functions $V \in C^{2,1}(R^n \times [-\tau, \infty); R_+)$ and $U \in C(R^n \times [-\tau, \infty); R_+)$, two probability measures $\mu(\cdot)$ and $\bar{\mu}(\cdot)$ on $[-\tau, 0]$ as well as a positive constant K, such that

$$\lim_{|x|\to\infty} \inf_{0\le t<\infty} V(x, t) = \infty, \tag{8}$$

while for all $(\varphi, t) \in C([-\tau, 0]; R^n) \times R_+$,

$$\begin{aligned}\mathcal{L}V(\varphi, t) &\le K\left(1 + V(\varphi(0), t) + \int_{-\tau}^{0} V(\varphi(\theta), t + \theta)\mathrm{d}\bar{\mu}(\theta)\right) \\ &\quad - U(\varphi(0), t) + \int_{-\tau}^{0} U(\varphi(\theta), t + \theta)\mathrm{d}\mu(\theta).\end{aligned} \tag{9}$$

We can now state our new existence-and-uniqueness theorem.

Theorem 2. *Let Assumptions 1 and 3 hold. Then for any given initial data (7), there is a unique global solution $x(t)$ to Eq. (6) on $t \in [-\tau, \infty)$. Moreover, the solution has the property that*

$$EV(x(t), t) \leq (0.5 + C)e^{Kt} - 0.5, \quad \forall t > 0, \tag{10}$$

where $C = V(x(0), 0) + K\int_{-\tau}^{0} V(x(s), s)\mathrm{d}s + \int_{-\tau}^{0} U(x(s), s)\mathrm{d}s$.

As the main aim of this paper is to establish new criteria on asymptotic stability and boundedness, we defer the proof to the Appendix. However, we would like to emphasise that the conditions in our theorem above are in terms of a pair of Lyapunov functions V and U and our theorem is a generalisation of the classical Khasminskii test (Khasminskii, 1980) on non-explosion for stochastic differential equations which is in terms of a single Lyapunov function.

3. Asymptotic stability and boundedness

With the notations introduced in the previous section, we can now state one of our main results.

Theorem 3. *Let Assumptions 1 and 3 hold except (9) which is replaced by*

$$\mathcal{L}V(\varphi, t) \leq \alpha_1 - \alpha_2 V(\varphi(0), t) + \alpha_3 \int_{-\tau}^{0} V(\varphi(\theta), t + \theta)\mathrm{d}\bar{\mu}(\theta) - U(\varphi(0), t) + \alpha \int_{-\tau}^{0} U(\varphi(\theta), t + \theta)\mathrm{d}\mu(\theta), \tag{11}$$

where $\alpha_1 \geq 0, \alpha_2 > \alpha_3 \geq 0$ and $\alpha \in (0, 1)$. Then for any given initial data (7), the unique global solution $x(t)$ to Eq. (6) has the property that

$$\limsup_{t\to\infty} EV(x(t), t) \leq \frac{\alpha_1}{\varepsilon}, \tag{12}$$

where $\varepsilon = \varepsilon_1 \wedge \varepsilon_2$ while $\varepsilon_2 = -\log(\alpha)/\tau$ and $\varepsilon_1 > 0$ is the unique root to the following equation

$$\alpha_2 = \varepsilon_1 + \alpha_3 e^{\varepsilon_1 \tau}. \tag{13}$$

If moreover $\alpha_1 = 0$, then

$$\limsup_{t\to\infty} \frac{1}{t}\log(EV(x(t), t)) \leq -\varepsilon \tag{14}$$

and

$$\int_0^\infty EU(x(t), t)\mathrm{d}t < \infty. \tag{15}$$

Proof. We first observe that (11) is stronger than (9). So, by Theorem 2, for any given initial data (7), Eq. (6) has a unique global solution $x(t)$ on $t \geq -\tau$. Let $k_0 > 0$ be sufficiently large for $\|\xi\| < k_0$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \geq 0 : |x(t)| \geq k\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). By the Itô formula and condition (11), we compute that, for $t \geq 0$,

$$\begin{aligned}
&E\left(e^{\varepsilon(t\wedge\tau_k)}V(x(t\wedge\tau_k), t\wedge\tau_k)\right) - V(x(0), 0) \\
&\quad = E\int_0^{t\wedge\tau_k} e^{\varepsilon s}\left(\varepsilon V(x(s), s) + \mathcal{L}V(x_s, s)\right)\mathrm{d}s \\
&\quad \leq \frac{\alpha_1 e^{\varepsilon t}}{\varepsilon} - (\alpha_2 - \varepsilon)E\int_0^{t\wedge\tau_k} e^{\varepsilon s}V(x(s), s)\mathrm{d}s \\
&\qquad + \alpha_3 E\int_0^{t\wedge\tau_k} e^{\varepsilon s}\left(\int_{-\tau}^{0} V(x(s+\theta), s+\theta)\mathrm{d}\bar{\mu}(\theta)\right)\mathrm{d}s \\
&\qquad - E\int_0^{t\wedge\tau_k} e^{\varepsilon s}U(x(s), s)\mathrm{d}s + \alpha E \\
&\qquad \times \int_0^{t\wedge\tau_k} e^{\varepsilon s}\left(\int_{-\tau}^{0} U(x(s+\theta), s+\theta)\mathrm{d}\mu(\theta)\right)\mathrm{d}s.
\end{aligned} \tag{16}$$

But, by the Fubini theorem (see Loève (1963)), we compute

$$\begin{aligned}
&E\int_0^{t\wedge\tau_k} e^{\varepsilon s}\left(\int_{-\tau}^{0} V(x(s+\theta), s+\theta)\mathrm{d}\bar{\mu}(\theta)\right)\mathrm{d}s \\
&\quad = E\int_{-\tau}^{0}\left(\int_0^{t\wedge\tau_k} e^{\varepsilon s}V(x(s+\theta), s+\theta)\mathrm{d}s\right)\mathrm{d}\bar{\mu}(\theta) \\
&\quad \leq e^{\varepsilon\tau}E\int_{-\tau}^{0}\left(\int_0^{t\wedge\tau_k} e^{\varepsilon(s+\theta)}V(x(s+\theta), s+\theta)\mathrm{d}s\right)\mathrm{d}\bar{\mu}(\theta) \\
&\quad \leq e^{\varepsilon\tau}E\int_{-\tau}^{0}\left(\int_{-\tau}^{t\wedge\tau_k} e^{\varepsilon s}V(x(s), s)\mathrm{d}s\right)\mathrm{d}\bar{\mu}(\theta) \\
&\quad = e^{\varepsilon\tau}E\int_{-\tau}^{t\wedge\tau_k} e^{\varepsilon s}V(x(s), s)\mathrm{d}s \\
&\quad \leq e^{\varepsilon\tau}\int_{-\tau}^{0} V(x(s), s)\mathrm{d}s + e^{\varepsilon\tau}E\int_0^{t\wedge\tau_k} e^{\varepsilon s}V(x(s), s)\mathrm{d}s.
\end{aligned}$$

Similarly

$$\begin{aligned}
&E\int_0^{t\wedge\tau_k} e^{\varepsilon s}\left(\int_{-\tau}^{0} U(x(s+\theta), s+\theta)\mathrm{d}\mu(\theta)\right)\mathrm{d}s \\
&\quad \leq e^{\varepsilon\tau}\int_{-\tau}^{0} U(x(s), s)\mathrm{d}s + e^{\varepsilon\tau}E\int_0^{t\wedge\tau_k} e^{\varepsilon s}U(x(s), s)\mathrm{d}s.
\end{aligned}$$

Substituting these into (16) gives

$$\begin{aligned}
&E\left(e^{\varepsilon(t\wedge\tau_k)}V(x(t\wedge\tau_k), t\wedge\tau_k)\right) \\
&\quad \leq C_1 + \frac{\alpha_1 e^{\varepsilon t}}{\varepsilon} - (\alpha_2 - \varepsilon - \alpha_3 e^{\varepsilon\tau})E \\
&\qquad \times \int_0^{t\wedge\tau_K} e^{\varepsilon s}V(x(s), s)\mathrm{d}s \\
&\qquad - (1 - \alpha e^{\varepsilon\tau})E\int_0^{t\wedge\tau_k} e^{\varepsilon s}U(x(s), s)\mathrm{d}s,
\end{aligned}$$

where $C_1 = V(x(0), 0) + \alpha_3 e^{\varepsilon\tau}\int_{-\tau}^{0} V(x(s), s)\mathrm{d}s + \alpha e^{\varepsilon\tau}\int_{-\tau}^{0} U(x(s), s)\mathrm{d}s$. On the other hand, for $\varepsilon \leq \varepsilon_1$ and $\varepsilon \leq \varepsilon_2$, we note from the definitions of ε_1 and ε_2 that

$$\alpha_2 - \varepsilon - \alpha_3 e^{\varepsilon\tau} \geq 0 \quad \text{and} \quad 1 - \alpha e^{\varepsilon\tau} \geq 0. \tag{17}$$

We therefore obtain

$$E\left(e^{\varepsilon(t\wedge\tau_k)}V(x(t\wedge\tau_k), t\wedge\tau_k)\right) \leq C_1 + \frac{\alpha_1 e^{\varepsilon t}}{\varepsilon}.$$

Letting $k \to \infty$ we obtain that

$$EV(x(t), t) \leq C_1 e^{-\varepsilon t} + \frac{\alpha_1}{\varepsilon}, \quad \forall t \geq 0, \tag{18}$$

and assertion (12) follows.

In the case when $\alpha_1 = 0$, (18) becomes

$$EV(x(t), t) \leq C_1 e^{-\varepsilon t}, \quad \forall t \geq 0$$

which yields assertion (14). To prove the other assertion (15), we can show by the Itô formula and condition (11) that

$$\begin{aligned}
0 &\leq V(x(0), 0) + \int_{-\tau}^{0}[\alpha_3 V(x(s), s) + \alpha U(x(s), s)]\mathrm{d}s \\
&\quad - (1-\alpha)E\int_0^t U(x(s), s)\mathrm{d}s.
\end{aligned}$$

This implies

$$E\int_0^t U(x(s),s)\mathrm{d}s \le \frac{1}{1-\alpha}\bigg(V(x(0),0) + \int_{-\tau}^{0}[\alpha_3 V(x(s),s)+\alpha U(x(s),s)]\mathrm{d}s\bigg)$$

for all $t \ge 0$. Applying the Fubini theorem and then letting $t \to \infty$ we obtain the desired assertion (15). □

Let us point out that the assertion $\int_0^\infty EU(x(t),t)\mathrm{d}t < \infty$ obtained in the theorem above is useful. For example, if we further have $U(x,t) \ge c|x|^2$ for some positive constant c, then this assertion implies that $\int_0^\infty E|x(t)|^2\mathrm{d}t < \infty$, which is known as the H_∞-stability.

Theorem 4. *Let all the assumptions of Theorem 3 hold and $\alpha_1 = 0$. Then for any given initial data (7), the unique global solution $x(t)$ to Eq. (6) has the property that*

$$\lim_{t\to\infty}\frac{1}{t}\log(V(x(t),t)) \le -\varepsilon \quad a.s. \tag{19}$$

where $\varepsilon > 0$ is the same as defined in Theorem 3.

Proof. The Itô formula shows that for any $t \ge 0$,

$$\mathrm{e}^{\varepsilon t}V(x(t),t) = V(x(0),0) + \int_0^t \mathrm{e}^{\varepsilon s}\left[\varepsilon V(x(s),s)+\mathcal{L}V(x_s,s)\right]\mathrm{d}s + M(t), \tag{20}$$

where $M(t) = \int_0^t \mathrm{e}^{\varepsilon s}V_x(x(s),s)g(x_s,s)\mathrm{d}B(s)$, which is a local martingale with the initial value $M(0) = 0$. By condition (11) with $\alpha_1 = 0$, we have

$$\begin{aligned}
&\int_0^t \mathrm{e}^{\varepsilon s}\left[\varepsilon V(x(s),s)+\mathcal{L}V(x_s,s)\right]\mathrm{d}s \\
&\le -(\alpha_2-\varepsilon)\int_0^t \mathrm{e}^{\varepsilon s}V(x(s),s)\mathrm{d}s \\
&\quad + \alpha_3\int_0^t \mathrm{e}^{\varepsilon s}\left(\int_{-\tau}^0 V(x(s+\theta),s+\theta)\mathrm{d}\bar{\mu}(\theta)\right)\mathrm{d}s \\
&\quad - \int_0^t \mathrm{e}^{\varepsilon s}U(x(s),s)\mathrm{d}s \\
&\quad + \alpha\int_0^t \mathrm{e}^{\varepsilon s}\left(\int_{-\tau}^0 U(x(s+\theta),s+\theta)\mathrm{d}\mu(\theta)\right)\mathrm{d}s.
\end{aligned}$$

But, in the same way as we did in the proof of Theorem 3, we can show that

$$\begin{aligned}
&\int_0^t \mathrm{e}^{\varepsilon s}\left(\int_{-\tau}^0 V(x(s+\theta),s+\theta)\mathrm{d}\bar{\mu}(\theta)\right)\mathrm{d}s \\
&\le \mathrm{e}^{\varepsilon\tau}\int_{-\tau}^0 V(x(s),s)\mathrm{d}s + \mathrm{e}^{\varepsilon\tau}\int_0^t \mathrm{e}^{\varepsilon s}V(x(s),s)\mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t \mathrm{e}^{\varepsilon s}\left(\int_{-\tau}^0 U(x(s+\theta),s+\theta)\mathrm{d}\mu(\theta)\right)\mathrm{d}s \\
&\le \mathrm{e}^{\varepsilon\tau}\int_{-\tau}^0 U(x(s),s)\mathrm{d}s + \mathrm{e}^{\varepsilon\tau}\int_0^t \mathrm{e}^{\varepsilon s}U(x(s),s)\mathrm{d}s.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_0^t \mathrm{e}^{\varepsilon s}\left[\varepsilon V(x(s),s)+\mathcal{L}V(x_s,s)\right]\mathrm{d}s \\
&\le \alpha_3\mathrm{e}^{\varepsilon\tau}\int_{-\tau}^0 V(x(s),s)\mathrm{d}s \\
&\quad - (\alpha_2-\varepsilon-\alpha_3\mathrm{e}^{\varepsilon\tau})\int_0^t \mathrm{e}^{\varepsilon s}V(x(s),s)\mathrm{d}s + \alpha\mathrm{e}^{\varepsilon\tau} \\
&\quad \times \int_{-\tau}^0 U(x(s),s)\mathrm{d}s - (1-\sigma\mathrm{e}^{\varepsilon\tau})\int_0^t \mathrm{e}^{\varepsilon s}U(x(s),s)\mathrm{d}s.
\end{aligned}$$

Recalling (17) we see that

$$\begin{aligned}
&\int_0^t \mathrm{e}^{\varepsilon s}\left[\varepsilon V(x(s),s)+\mathcal{L}V(x_s,s)\right]\mathrm{d}s \\
&\le \mathrm{e}^{\varepsilon\tau}\int_{-\tau}^0[\alpha_3 V(x(s),s)+\alpha U(x(s),s)]\mathrm{d}s.
\end{aligned}$$

Substituting this into (20) we get

$$\begin{aligned}
&\mathrm{e}^{\varepsilon t}V(x(t),t) \le V(x(0),0) \\
&\quad + \mathrm{e}^{\varepsilon\tau}\int_{-\tau}^0[\alpha_3 V(x(s),s)+\alpha U(x(s),s)]\mathrm{d}s + M(t).
\end{aligned} \tag{21}$$

Applying the non-negative semi-martingale convergence theorem (see e.g. Lipster and Shiryayev (1989, Theorem 7 on page 139) or Mao and Yuan (2006, Theorem 1.10 on page 18)), we obtain that

$$\limsup_{t\to\infty}\left[\mathrm{e}^{\varepsilon t}V(x(t),t)\right] < \infty \quad a.s.$$

Hence, there is a finite positive random variable ζ such that

$$\sup_{0\le t<\infty}\left[\mathrm{e}^{\varepsilon t}V(x(t),t)\right] \le \zeta \quad a.s.$$

This implies

$$\limsup_{t\to\infty}\frac{1}{t}\log(V(x(t),t)) \le -\varepsilon \quad a.s.$$

as required. □

Two theorems above give asymptotic estimates on $EV(x(t),t)$ or $V(x(t),t)$. With a little bit more information on $V(x,t)$, e.g. $V(x,t) \ge c|x|^p$, we can obtain asymptotic estimates on $x(t)$. We state these results as a corollary.

Corollary 1. *Let the assumptions of Theorem 3 hold. If there is moreover a pair of positive constants c and p such that*

$$c|x|^p \le V(x,t) \quad \forall (x,t) \in R^n \times [-\tau,\infty).$$

Then for any given initial data (7), the unique global solution $x(t)$ to Eq. (6) obeys

$$\limsup_{t\to\infty} E|x(t)|^p \le \frac{\alpha_1}{\varepsilon c}, \tag{22}$$

where $\varepsilon > 0$ is the same as defined in Theorem 3. If, furthermore, $\alpha_1 = 0$, then

$$\limsup_{t\to\infty}\frac{1}{t}\log(E|x(t)|^p) \le -\varepsilon \tag{23}$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t)|) \le -\frac{\varepsilon}{p} \quad a.s. \tag{24}$$

The following theorem estimates the limit of the average in time of the moment.

Theorem 5. *Let Assumptions 1 and 3 hold except (9) which is replaced by*

$$\mathcal{L}V(\varphi,t)\le \alpha_1-\alpha_2 V(\varphi(0),t)+\alpha_2\int_{-\tau}^{0}V(\varphi(\theta),t+\theta)\mathrm{d}\bar{\mu}(\theta)$$
$$-U(\varphi(0),t)+\alpha\int_{-\tau}^{0}U(\varphi(\theta),t+\theta)\mathrm{d}\mu(\theta), \tag{25}$$

where $\alpha_1,\alpha_2\ge 0$ and $\alpha\in(0,1)$. Then for any given initial data (7), the unique global solution $x(t)$ to Eq. (6) has the property that

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t EU(x(s),s)\mathrm{d}s\le\frac{\alpha_1}{1-\alpha}. \tag{26}$$

In particular, if there is moreover a pair of positive constants c and p such that

$$c|x|^p\le U(x,t)\quad \forall (x,t)\in R^n\times[-\tau,\infty), \tag{27}$$

then

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t E|x(s)|^p\mathrm{d}s\le\frac{\alpha_1}{c(1-\alpha)}. \tag{28}$$

Proof. We once again observe that (25) is stronger than (9). So, by Theorem 2, for any given initial data (7), Eq. (6) has a unique global solution $x(t)$ on $t\ge -\tau$. Let τ_k be the same stopping time defined in the proof of Theorem 3. For any $t\ge 0$, by the Itô formula and condition (25), we compute that

$$\begin{aligned}EV(x(t\wedge\tau_k),t\wedge\tau_k)&\le V(x(0),0)+\alpha_1 t-\alpha_2 E\int_0^{t\wedge\tau_k}V(x(s),s)\mathrm{d}s\\&-\alpha_2E\int_0^{t\wedge\tau_k}\left(\int_{-\tau}^0 V(x(s+\theta),s+\theta)\mathrm{d}\bar{\mu}(\theta)\right)\mathrm{d}s\\&-E\int_0^{t\wedge\tau_k}U(x(s),s)\mathrm{d}s\\&+\alpha E\int_0^{t\wedge\tau_k}\left(\int_{-\tau}^0 U(x(s+\theta),s+\theta)\mathrm{d}\mu(\theta)\right)\mathrm{d}s.\end{aligned} \tag{29}$$

But, by the Fubini theorem, we compute

$$\begin{aligned}&\int_0^{t\wedge\tau_k}\left(\int_{-\tau}^0 U(x(s+\theta),s+\theta)\mathrm{d}\mu(\theta)\right)\mathrm{d}s\\&=\int_{-\tau}^0\left(\int_0^{t\wedge\tau_k}U(x(s+\theta),s+\theta)\mathrm{d}s\right)\mathrm{d}\mu(\theta)\\&\le\int_{-\tau}^0\left(\int_{-\tau}^{t\wedge\tau_k}U(x(s),s)\mathrm{d}s\right)\mathrm{d}\mu(\theta)\\&\le\int_{-\tau}^{t\wedge\tau_k}U(x(s),s)\mathrm{d}s.\end{aligned} \tag{30}$$

Similarly,

$$\begin{aligned}&\int_0^{t\wedge\tau_k}\left(\int_{-\tau}^0 V(x(s+\theta),s+\theta)\mathrm{d}\bar{\mu}(\theta)\right)\mathrm{d}s\\&\le\int_{-\tau}^{t\wedge\tau_k}V(x(s),s)\mathrm{d}s.\end{aligned} \tag{31}$$

Substituting these into (29), we get

$$\begin{aligned}&EV(x(t\wedge\tau_k),t\wedge\tau_k)\\&\le\bar{C}+\alpha_1 t-(1-\alpha)E\int_0^{t\wedge\tau_k}U(x(s),s)\mathrm{d}s,\end{aligned} \tag{32}$$

where $\bar{C}=V(x(0),0)+\int_{-\tau}^0[\alpha_2V(x(s),s)\mathrm{d}s+\alpha U(x(s),s)]\mathrm{d}s$. Consequently

$$(1-\alpha)E\int_0^{t\wedge\tau_k}U(x(s),s)\mathrm{d}s\le\bar{C}+\alpha_1 t.$$

Letting $k\to\infty$ and then by the Fubini theorem we get

$$(1-\alpha)\int_0^t EU(x(s),s)\mathrm{d}s\le\bar{C}+\alpha_1 t.$$

This implies the required assertion (26), which yields the other assertion (28) if the additional condition (27) is fulfilled. □

If we compare our new results with the known result, Theorem 1, we see the following significant improvements:

- The linear growth condition on the coefficients f and g is no longer required.
- The bound for $\mathcal{L}V$ is in a much weaker form.
- Our new results do not only deal with the asymptotic moment estimation but also the path-wise (almost sure) estimation.

4. Examples

Let us discuss a number of examples to illustrate these advantages. In the following examples, we let $B(t)$ be a scalar Brownian motion.

Example 1. Let us first return to the SFDE (2), where D and $\bar{D}$ obey (3) and (4), respectively, with $K=\bar{K}=1$. Recalling (5), we observe that condition (11) is fulfilled with $V(x,t)=x^2$, $U(x,t)=2x^4$, $\alpha_1=0$, $\alpha_2=\frac{7}{5}$, $\alpha_3=1$, $\alpha=\frac{5}{6}$. Set $\varepsilon=\varepsilon_1\wedge\varepsilon_2$, where $\varepsilon_2=\log(1.2)/\tau$ and $\varepsilon_1>0$ is the unique root to the equation $\frac{7}{5}=\varepsilon_1+\mathrm{e}^{\varepsilon_1\tau}$. By Theorems 3 and 4, we can conclude that the solution of the SFDE (2) has the following properties that

$$\limsup_{t\to\infty}\frac{1}{t}\log(E|x(t)|^2)\le-\varepsilon,$$

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t)|)\le-\frac{\varepsilon}{2}\quad a.s.$$

and

$$\int_0^\infty E|x(t)|^4\mathrm{d}t<\infty.$$

Example 2. Consider a scalar SFDE with an additive noise of the form

$$\mathrm{d}x(t)=[-x^3(t)+(D(x_t))^2]\mathrm{d}t+\sigma\mathrm{d}B(t) \tag{33}$$

on $t\ge 0$ with initial data $x_0=\xi\in C([-\tau,0];R)$. We assume that D obeys (3) (but we do not ask $K=1$ anymore). We claim that for any integer $p\ge 1$, the solution obeys

$$\limsup_{t\to\infty}E|x(t)|^{2p}<\infty. \tag{34}$$

Let $V(x,t)=x^{2p}$ and compute

$$\begin{aligned}\mathcal{L}V(\varphi,t)&=2p\varphi^{2p-1}(0)[-\varphi^3(0)+(D(\varphi))^2]\\&+p(2p-1)\sigma^2\varphi^{2p-2}(0).\end{aligned} \tag{35}$$

By the Young inequality

$$u^\alpha v^{1-\alpha}\le\alpha u+(1-\alpha)v,\quad\forall u,v\ge 0,\alpha\in(0,1),$$

we can show that

$$\varphi^{2p-1}(0)(D(\varphi))^2\le|\varphi(0)|^{2p+1}+K^{2p+1}\int_{-\tau}^0|\varphi(\theta)|^{2p+1}\mathrm{d}\mu(\theta).$$

It then follows from (35) that

$$\mathcal{L}V(\varphi,t) \le \kappa(\varphi(0)) - |\varphi(0)|^{2p} - (2pK^{2p+1}+1)|\varphi(0)|^{2p+1} + 2pK^{2p+1}\int_{-\tau}^{0}|\varphi(\theta)|^{2p+1}\mathrm{d}\mu(\theta), \qquad (36)$$

where $\kappa(z) = -2p|z|^{2p+2} + (2pK^{2p+1}+2p+1)|z|^{2p+1} + |z|^{2p} + p(2p-1)\sigma^2|z|^{2p-2}$ for $z \in R$. Clearly $\kappa(\cdot)$ is bounded above, say by $\hat{\kappa}$, in R. We hence have

$$\mathcal{L}V(\varphi,t) \le \hat{\kappa} - |\varphi(0)|^{2p} - (2pK^{2p+1}+1)|\varphi(0)|^{2p+1} + 2pK^{2p+1}\int_{-\tau}^{0}|\varphi(\theta)|^{2p+1}\mathrm{d}\mu(\theta). \qquad (37)$$

Hence, condition (11) is fulfilled with $U(x,t) = (2pK^{2p+1}+1)|x|^{2p+1}$, $\alpha_1 = \hat{\kappa}$, $\alpha_2 = 1$, $\alpha_3 = 0$, $\alpha = \frac{2pK^{2p+1}}{2pK^{2p+1}+1}$. The desired result (34) follows from Theorem 3.

Example 3. Let us finally consider a 2-dimensional SFDE

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left([a_{11} - a_{12}x_1^2(t) + a_{13}D(x_{2,t})]\mathrm{d}t \right.\\ \qquad \left. +\ a_{14}\sqrt{D(x_{2,t})}\mathrm{d}B(t)\right), \\ \mathrm{d}x_2(t) = x_2(t)\left([a_{21} - a_{22}x_2^2(t) + a_{23}D(x_{1,t})]\mathrm{d}t \right.\\ \qquad \left. +\ a_{24}\sqrt{D(x_{1,t})}\mathrm{d}B(t)\right), \end{cases} \qquad (38)$$

where a_{ij}'s are all positive constants, $x_{1,t} = \{x_1(t+\theta) : -\tau \le \theta \le 0\}$, $x_{2,t} = \{x_2(t+\theta) : -\tau \le \theta \le 0\}$ and $D : C([-\tau,0); R) \to R$ is defined by

$$D(\phi) = \frac{1}{\tau}\int_{-\tau}^{0}|\phi(\theta)|\mathrm{d}\theta.$$

Such SFDEs have been used to model population systems under noise and are known as the stochastic power law logistic model (see Bahar and Mao (2008) and the references therein). Let $V(x,t) = x^2$. Then the corresponding functional $\mathcal{L}V : C([-\tau,0]; R^2) \times R_+ \to R$ has the form

$$\begin{aligned}\mathcal{L}V(\varphi,t) &= 2\varphi_1^2(0)[a_{11} - a_{12}\varphi_1^2(0) + a_{13}D(\varphi_2)] \\ &\quad + 2\varphi_2^2(0)[a_{21} - a_{22}\varphi_2^2(0) + a_{23}D(\varphi_1)] \\ &\quad + a_{14}^2\varphi_1^2(0)D(\varphi_2) + a_{24}^2\varphi_2^2(0)D(\varphi_1),\end{aligned}$$

where $\varphi = (\varphi_1, \varphi_2)^T \in C([-\tau,0]; R^2)$. By the Young inequality (stated in Example 2), we can show

$$\varphi_1^2(0)D(\varphi_2) \le |\varphi_1(0)|^3 + \frac{1}{\tau}\int_{-\tau}^{0}|\varphi_2(\theta)|^3\mathrm{d}\theta$$

and

$$\varphi_2^2(0)D(\varphi_1) \le |\varphi_2(0)|^3 + \frac{1}{\tau}\int_{-\tau}^{0}|\varphi_1(\theta)|^3\mathrm{d}\theta.$$

Hence

$$\begin{aligned}\mathcal{L}V(\varphi,t) &\le 2\varphi_1^2(0)[a_{11} - a_{12}\varphi_1^2(0)] \\ &\quad + (2a_{13} + a_{14}^2)\left(|\varphi_1(0)|^3 + \frac{1}{\tau}\int_{-\tau}^{0}|\varphi_2(\theta)|^3\mathrm{d}\theta\right) \\ &\quad + 2\varphi_2^2(0)[a_{21} - a_{22}\varphi_2^2(0)] \\ &\quad + (2a_{23} + a_{24}^2)\left(|\varphi_2(0)|^3 + \frac{1}{\tau}\int_{-\tau}^{0}|\varphi_1(\theta)|^3\mathrm{d}\theta\right) \\ &\le \kappa(\varphi(0)) - |\varphi(0)|^2 - U(\varphi(0)) + \frac{1}{2\tau}\int_{-\tau}^{0}U(\varphi(\theta))\mathrm{d}\theta,\end{aligned}$$

where $U(x) = 2\bar{a}(|x_1|^3 + |x_2|^3)$ for $x \in R^2$ with $\bar{a} = (2a_{13} + a_{14}^2) \vee (2a_{23} + a_{24}^2)$ and

$$\kappa(x) = 2x_1^2[a_{11} - a_{12}x_1^2] + 2x_2^2[a_{21} - a_{22}x_2^2] + |x|^2 + 1.5U(x).$$

Clearly, $\kappa(x)$ is bounded above in $x \in R^2$, namely $\bar{k} := \sup_{x\in R^2}\kappa(x) < \infty$. Consequently $\mathcal{L}V(\varphi,t) \le \bar{\kappa} - |\varphi(0)|^2 - U(\varphi(0)) + \frac{1}{2\tau}\int_{-\tau}^{0}U(\varphi(\theta))\mathrm{d}\theta$. By Theorem 3, we see that the solution of the SFDE (38) obeys

$$\limsup_{t\to\infty} E|x(t)|^2 \le \bar{k}. \qquad (39)$$

To apply Theorem 4, we note that

$$|x|^3 \le (|x_1| + |x_2|)^3 \le 4(|x_1|^3 + |x_2|^3) = \frac{2}{\bar{a}}U(x), \quad x \in R^2,$$

whence we can conclude that the solution also obeys

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t E|x(t)|^3\mathrm{d}t \le \frac{4\bar{k}}{\bar{a}}. \qquad (40)$$

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Appendix

In this appendix we shall prove Theorem 2. It is known that the SFDE may only have a local solution without the linear growth condition, that is an explosion may occur at a finite time. As the local solution will play a key role in this proof, we cite the definition from Mao (2007, Definition 2.7 on page 154).

Definition 1. Let $x(t)$, $-\tau \le t < \sigma_\infty$, be a continuous $\mathcal{F}_t$-adapted R^n-valued local process, where σ_∞ is a stopping time and we set $\mathcal{F}_t = \mathcal{F}_0$ for $t \in [-\tau, 0)$. It is called a local solution of Eq. (6) with initial data (7) if $x_0 = \xi$ and for all $t \ge 0$

$$x(t\wedge\sigma_k) = \xi(0) + \int_0^{t\wedge\sigma_k} f(x_s,s)\mathrm{d}s + \int_0^{t\wedge\sigma_k} g(x_s,s)\mathrm{d}B(s)$$

holds for any $k \ge 1$, where $\{\sigma_k\}_{k\ge1}$ is a non-decreasing sequence of finite stopping times such that $\sigma_k \uparrow \sigma_\infty$ a.s. Furthermore, if $\limsup_{k\to\infty}|x(\sigma_k)| = \infty$ is satisfied whenever $\sigma_\infty < \infty$, it is called a maximal local solution and σ_∞ is called the explosion time. A maximal local solution $x(t)$, $-\tau \le t < \sigma_\infty$, is said to be unique if for any other maximal local solution $\hat{x}(t)$, $-\tau \le t < \hat{\sigma}_\infty$, we have $\sigma_\infty = \hat{\sigma}_\infty$ a.s. and $x(t) = \hat{x}(t)$ for all $-\tau \le t < \sigma_\infty$ a.s.

Proof of Theorem 2. By Mao (2007, Theorem 2.8 on page 154), there is a unique maximal local solution $x(t)$ on $t \in [-\tau, \sigma_\infty)$, where σ_∞ is the explosion time. Let $k_0 > 0$ be sufficiently large for $\|\xi\| < k_0$. For each integer $k \ge k_0$, define the stopping time

$$\tau_k = \inf\{t \in [0, \sigma_\infty) : |x(t)| \ge k\}.$$

Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty}\tau_k$, whence $\tau_\infty \le \sigma_\infty$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\sigma_\infty = \infty$ a.s. We need to show $\tau_\infty = \infty$ a.s. and assertion (10). By the Itô formula, for any $k \ge k_0$ and $t \ge 0$,

$$EV(x(t\wedge\tau_k), t\wedge\tau_k) = V(x(0),0) + E\int_0^{t\wedge\tau_k}\mathcal{L}V(x_s,s)\mathrm{d}s.$$

By condition (9), we therefore have

$$\begin{aligned} &EV(x(t\wedge\tau_k), t\wedge\tau_k) \\ &\quad \le V(x(0),0) + E\int_0^{t\wedge\tau_k} K\Bigg(1 + V(x(s),s) \\ &\qquad + \int_{-\tau}^0 V(x(s+\theta), \sigma+\theta)\mathrm{d}\bar{\mu}(\theta)\Bigg)\mathrm{d}s \\ &\qquad + E\int_0^{t\wedge\tau_k}\Bigg(-U(x(s),s) \\ &\qquad + \int_{-\tau}^0 U(x(s+\theta), s+\theta)\mathrm{d}\mu(\theta)\Bigg)\mathrm{d}s. \end{aligned} \tag{A.1}$$

Recalling (30) and (31), we then have

$$EV(x(t\wedge\tau_k), t\wedge\tau_k) \le C + E\int_0^{t\wedge\tau_k} K(1 + 2V(x(s),s))\mathrm{d}s, \tag{A.2}$$

where C has been defined in the statement of the theorem. It then follows from (A.2) that

$$\begin{aligned} &0.5 + EV(x(t\wedge\tau_k), t\wedge\tau_k) \\ &\quad \le 0.5 + C + E\int_0^{t\wedge\tau_k} 2K(0.5 + V(x(s),s))\mathrm{d}s \\ &\quad = 0.5 + C + E\int_0^{t\wedge\tau_k} 2K(0.5 + V(x(s\wedge\tau_k), s\wedge\tau_k))\mathrm{d}s \\ &\quad = 0.5 + C + \int_0^{t} 2K\,(0.5 + E(V(x(s\wedge\tau_k), s\wedge\tau_k)))\,\mathrm{d}s. \end{aligned}$$

The Gronwall inequality shows that

$$0.5 + EV(x(t\wedge\tau_k), t\wedge\tau_k) \le (0.5 + C)\mathrm{e}^{2Kt},$$

whence

$$EV(x(t\wedge\tau_k), t\wedge\tau_k) \le (0.5 + C)\mathrm{e}^{2Kt} - 0.5. \tag{A.3}$$

Define

$$\upsilon_k = \inf_{|x|\ge k, 0\le t<\infty} V(x,t) \quad \text{for } k \ge k_0.$$

Compute

$$\begin{aligned} EV(x(t\wedge\tau_k), t\wedge\tau_k) &\ge E\left(V(x(\tau_k), \tau_k) I_{\{\tau_k\le T\}}\right) \\ &\ge \upsilon_k P(\tau_k \le T), \end{aligned}$$

where throughout this paper I_G denotes the indicator function of set G. It then follows from (A.3) that

$$\upsilon_k P(\tau_k \le t) \le (0.5 + C)\mathrm{e}^{Kt} - 0.5.$$

But, by condition (8), $\lim_{k\to\infty}\upsilon_k = \infty$. Letting $\kappa \to \infty$ in the inequality above, we then see $P(\tau_\infty \le t) = 0$. Since $t > 0$ is arbitrary, we must have that $P(\tau_\infty < \infty) = 0$, whence $\tau_\infty = \infty$ a.s. Finally, letting $k \to \infty$ in (A.3) yields the required assertion (10). The proof is therefore complete. □

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