Comment on “Application of \( (G'/G) \)-expansion method to travelling wave solutions of three nonlinear evolution equation” [Computers & Fluids 2010;39:1957–1963]

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Abstract

In a recent paper [Abazari R. Application of \((G'/G)\)-expansion method to travelling wave solutions of three nonlinear evolution equation. Computers & Fluids 2010;39:1957–1963], the \((G'/G)\)-expansion method was used to find travelling-wave solutions to three nonlinear evolution equations that arise in the mathematical modelling of fluids. The author claimed that the method delivers more general forms of solution than other methods. In this note we point out that not only is this claim false but that the delivered solutions are cumbersome and misleading. The extended tanh-function expansion method, for example, is not only entirely equivalent to the \((G'/G)\)-expansion method but is more efficient and user-friendly, and delivers solutions in a compact and elegant form.

Key words: Nonlinear evolution equations; Travelling-wave solutions; \((G'/G)\)-expansion method; Tanh-function expansion method.

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Over the past two decades or so several methods for finding travelling-wave solutions to nonlinear evolution equations have been proposed, developed and extended. The solutions to dozens of equations have been found by one or other of these methods. References [1–12] and some of the references therein mention some of this activity.

One recent method that has proved to be popular is the \((G'/G)\)-expansion method originally proposed by Wang et al. [1]. The solutions delivered by this method look rather cumbersome; furthermore, they appear to have more free parameters than solutions delivered by other methods. This has led to two unfortunate phenomena:

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(i) Some authors claim that the \((G'/G)\)-expansion method delivers solutions that are ‘more general’ than those delivered by other methods;

(ii) Some authors go on to claim that they have found ‘new’ solutions when often the truth is that the so-called ‘new’ solutions are merely disguised versions of previously known solutions that may be found by more efficient methods.

The claim in phenomenon (i) was shown to be false independently in three recent papers [2–4]; in each of these papers it is shown that the \((G'/G)\)-expansion method is entirely equivalent to the extended tanh-function expansion method originally proposed by Fan [5]. Furthermore, in [4] we showed that the latter method is more efficient and more user-friendly than the former method. This was illustrated with reference to the KdV equation.

To illustrate both phenomena we focus on a paper that appeared recently in this journal [6]. In [6], the \((G'/G)\)-expansion method was applied to three nonlinear evolution equations that arise in the mathematical modelling of fluids. For illustrative purposes we consider the solutions only for the first of these equations, namely the transformed reduced Ostrovsky equation (dubbed the Vakhnenko–Parkes equation in [6])

\[ uu_{xt} - u_x u_{xt} + u^2 u_t = 0; \]  \hspace{1cm} (1)

Similar observations may be made about the solutions to the other two equations, namely the regularized long wave (RLW) equation and the symmetric RLW equation.

In [7] we summarized the derivation of nine solutions to Eq. (1) that are delivered efficiently by the extended tanh-function expansion method, namely

\begin{align*}
    u_{11} &= 6K^2 \text{sech}^2(K\eta), \\
    u_{12} &= -6K^2 \text{cosech}^2(K\eta), \\
    u_{13} &= -6K^2 \sec^2(K\eta), \\
    u_{14} &= -6K^2 \text{cosec}^2(K\eta), \\
    u_{21} &= -4K^2 + 6K^2 \text{sech}^2(k\eta), \\
    u_{22} &= -4K^2 - 6K^2 \text{cosech}^2(k\eta), \\
    u_{23} &= 4K^2 - 6K^2 \sec^2(K\eta), \\
    u_{24} &= 4K^2 - 6K^2 \text{cosec}^2(K\eta), \\
    u_3 &= -6/\eta^2,
\end{align*}

where \( \eta := x - ct - x_0 \), and \( K, c \) and \( x_0 \) are arbitrary real constants. (Note that use of a computer takes the drudgery out of applying expansion methods by hand. For example, the basic tanh-function method may be applied with minimal effort by use of the automated tanh-function method [8] which uses the Mathematica package ATFM.) In [7], we also described where these solutions have appeared before in the literature.
In [6] it is shown that the solution to Eq. (1) is given by

$$u(x, t) = -6k^2 \left[ \left( \frac{G'}{G} \right)^2 + \lambda \left( \frac{G'}{G} \right) + \mu \right],$$

(11)

where $k$, $\lambda$ and $\mu$ are arbitrary constants, and $(G'/G)$ has three different forms depending on whether $\lambda^2 - 4\mu$ is positive, negative or zero. For example, when $\lambda^2 - 4\mu > 0$,

$$\frac{G'((x, t))}{G((x, t))} = \sqrt{\lambda^2 - 4\mu} \left( C_1 \sinh \theta + C_2 \cosh \theta \right) - \frac{\lambda}{2},$$

(12)

where $\theta = \sqrt{\lambda^2 - 4\mu} \xi/2$, $\xi = kx + \omega t$, and $\omega$, $C_1$ and $C_2$ are arbitrary constants. In this case (11) becomes (18a) in [6], namely

$$u(x, t) = \frac{3k^2(\lambda^2 - 4\mu)(C_1^2 - C_2^2)}{2(C_2 \sinh \theta + C_1 \cosh \theta)^2}. $$

(13)

If $C_1^2 > C_2^2$, then (13) may be rewritten as

$$u(x, t) = \frac{3}{2} k^2(\lambda^2 - 4\mu) \text{sech}^2(\theta + \theta_0),$$

(14)

where $\text{tanh} \theta_0 := C_2/C_1$; if $C_1^2 < C_2^2$, then (13) may be rewritten as

$$u(x, t) = -\frac{3}{2} k^2(\lambda^2 - 4\mu) \text{cosech}^2(\theta + \theta_0),$$

(15)

where $\text{coth} \theta_0 := C_2/C_1$.

The solutions given by (14) and (15) are equivalent to (18b) and (18c) in [6]. Note that, apparently, these two solutions have six free parameters, namely $k$, $\omega$, $\lambda$, $\mu$, $C_1$ and $C_2$. However, if we introduce the quantities $c$, $K$ and $x_0$ defined by

$$c = -\frac{\omega}{k}, \quad K = \frac{k\sqrt{\lambda^2 - 4\mu}}{2}, \quad Kx_0 = -\theta_0,$$

(16)

then (14) and (15) reduce to $u_{11}$ and $u_{12}$, respectively. In this form the solutions have only three free parameters, namely $c$, $K$ and $x_0$. Similarly, with

$$c = -\frac{\omega}{k}, \quad K = \frac{k\sqrt{4\mu - \lambda^2}}{2}, \quad Kx_0 = -\theta_0,$$

(17)

the solutions given by (19b) and (19c) in [6] can be rewritten as $u_{13}$ and $u_{14}$, respectively. A fifth solution is given by (20) in [6]; apparently it has four free parameters. However, with $c = -\omega/k$ and $x_0 = -C_1/(kC_2)$, the solution reduces to $u_3$ which has only two free parameters, namely $c$ and $x_0$. Solutions corresponding to $u_{21}$, $u_{22}$, $u_{23}$ and $u_{24}$ were not presented in [6].
The foregoing illustrative discussion exemplifies two deficiencies of the \((G'/G)\)-expansion method: firstly, that the method delivers solutions in a cumbersome form (see (11) with (12), or (13), for example) and secondly, that the solutions appear to contain more free parameters than is actually the case. Typically, each solution can be manipulated into a neater form which displays the correct number of free parameters. These neat forms are the ones that are delivered directly and more efficiently by the extended tanh-function method.

It is of interest to note that the introduction of the parameter \(k\) into the \((G'/G)\)-expansion method as presented in [6] is superfluous. This embellishment is not present in the descriptions of the method in [1–4]. Furthermore, as has been pointed out in [9], the parameter \(\lambda\) is superfluous at least as far as the basic method is concerned. These observations are exemplified by setting \(k = 1\) and \(\lambda = 0\) in the above argument: the generality of the solutions is unaffected.

Finally, we make some additional pertinent comments regarding Eq. (1). In [10], the Exp-function method was used to derive two solutions to Eq. (1). These solutions are rather cumbersome but, as shown in [11], they may be reduced to \(u_{11}\) and \(u_{21}\) respectively. Reference to [10] is also made in [6]; in the latter it is shown that, by use of particular values of some of the parameters, (14) reduces to a special case of \(u_{11}\) which is also one of the special cases mentioned in [10]. In [12], an auxiliary equation method was used to solve Eq. (1). Twenty eight solutions were derived including ‘many new’ solutions. However, in [11] we showed that all twenty eight solutions may be reduced to \(u_{11}\) or \(u_{12}\). In [13], the solutions \(u_{11}, u_{21}\) and \(u_{3}\) were derived via a method involving Laurent series.

In [13], periodic-wave solutions to Eq. (1) in terms of the Jacobi elliptic \(cn\) function were derived by direct integration. These solutions may also be derived easily by the Jacobi elliptic-function expansion method as outlined in [14], for example. We have used the semi-automated procedure that we described in [14] to obtain the aforementioned solutions in a more user-friendly form, namely

\[
\begin{align*}
  u(x, t) &= 6mK^2 \text{cn}^2(K\eta|m) + 2 \left(1 - 2m \pm \sqrt{1 - m + m^2}\right)K^2, \\
&= \left(1 - \pm \sqrt{1 - m + m^2}\right)K^2, \quad (18)
\end{align*}
\]

where \(K\), \(c\) and \(x_0\) are arbitrary, and \(m\) is a parameter such that \(0 < m \leq 1\). In the limit \(m \to 1\), the two solutions given by (18) reduce straightforwardly to \(u_{11}\) and \(u_{21}\), respectively.

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References


