Similarity solutions for slender rivulets with thermocapillarity

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Abstract

We use the lubrication approximation to investigate the steady flow of slender non-uniform rivulets of a viscous fluid on an inclined plane that is either heated or cooled relative to the surrounding atmosphere. Four non-isothermal situations in which thermocapillary effects play a significant role are considered. We derive the general equations for a slender rivulet subject to gravity, surface tension, thermocapillarity and a constant surface shear stress. Similarity solutions describing a thermocapillary-driven rivulet widening or narrowing due to either gravitational or surface-tension effects on a non-uniformly heated or cooled substrate are obtained, and we present examples of these solutions when the substrate temperature gradient depends on the longitudinal coordinate according to a general power law. When gravitational effects are strong there is a unique solution representing both a narrowing pendent rivulet and a widening sessile rivulet whose transverse profile always has a single global maximum. When surface-tension effects are strong there is a one-parameter family of solutions representing both a narrowing and a widening rivulet whose transverse profile has either a single global maximum or two equal global maxima and a local minimum. Unique similarity solutions whose transverse profiles always have a single global maximum are also obtained for both a gravity-driven and a constant-surface-shear-stress-driven rivulet widening or narrowing due to thermocapillarity on a uniformly heated or cooled substrate. The solutions in both cases represent both a narrowing rivulet on a heated substrate and a widening rivulet on a cooled substrate (albeit with infinite width in the gravity-driven case).

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1 Introduction

Rivulets occur in a wide variety of practical situations ranging from heat exchangers, condensers and evaporators to industrial coating processes. In many situations (such as, for example, in many geophysical flows) gravity plays an important role, while in others (such as, for example, in the flow of rain-water on the windscreen of a moving car) the presence of an external air flow is significant. There are also situations, such as in microscopic fluidic devices (see, for example, Kataoka and Troian [1]), in which thermocapillary effects due to the variation of surface tension with temperature play an important role. In this paper we shall consider steady rivulet flows driven by gravity, thermocapillarity or a constant surface shear stress on a heated or cooled inclined plane when thermocapillary effects play a significant role.

Smith [2] obtained a similarity solution describing the steady gravity-driven draining of a slender non-uniform rivulet of viscous fluid from a point source or to a point sink on an inclined plane in the absence of surface-tension effects; subsequently Duffy and Moffatt [3] obtained the corresponding solution when surface-tension effects are dominant. Smith's [2] solution predicts that the width of the rivulet increases or decreases according to an $x^{3/7}$ power law and the height correspondingly decreases or increases like $x^{-1/7}$, where x is the longitudinal coordinate. These predictions are in excellent agreement with Smith's own experimental results and with the numerical results of Schwartz and Michaelides [4]. Duffy and Moffatt [3] found that when surface-tension effects are dominant, the power laws are modified to $x^{3/13}$ for the width and $x^{-1/13}$ for the height. Wilson, Duffy and Hunt [5] used the approach of Smith [2] and Duffy and Moffatt [3] to obtain the corresponding similarity solutions describing a slender non-uniform rivulet of a non-Newtonian power-law fluid driven by either gravity or a constant surface shear stress down an inclined plane. The same approach has also been used by Wilson, Duffy and Davis [6] to study the closely related problem of a slender dry patch in an infinitely wide thin film draining under gravity down an inclined plane.

The pioneering analysis of the steady unidirectional gravity-driven flow of a uniform rivulet of viscous fluid down an inclined plane was performed by Towell and Rothfeld [7] who calculated the profile of the rivulet numerically and found excellent agreement with their own experimental results. Allen and Biggin [8] and Duffy and Moffatt [9] used the lubrication approximation to obtain analytically the solution when the cross-sectional profile of the rivulet transverse to the direction of flow is thin. Duffy and Moffatt [9] also interpreted their results as describing the locally unidirectional flow down a locally planar substrate whose local slope varies slowly in the longitudinal direction. This approach was recently used by Holland, Duffy and Wilson [10] to investigate locally uniform (but not locally unidirectional) gravity-driven rivulet flow down a uniformly heated or cooled slowly varying substrate when thermocapillary effects play a significant role. In particular, they found that the variation in surface tension drives a transverse flow that causes the fluid particles to spiral down the rivulet in helical vortices.

The pioneering work on non-isothermal thin-film flow was performed by Burelbach, Bankoff and Davis [11] who included the effects of mass loss or gain, vapour recoil, thermocapillarity, surface tension, gravity and long-range intermolecular attraction in their analysis of a two-dimensional thin film of fluid on a uniformly heated or cooled horizontal substrate. This work laid the foundations for a large number of subsequent studies (see, for example, the discussion in [10] and the references therein) on a wide variety of steady and unsteady non-isothermal thin-film flows. In addition there have also been various studies of other thin-film flows in which the surface shear stress and/or pressure gradient due to an external airflow are significant (see, for example, the discussion in [5] and the references therein). Oron, Davis and Bankoff [12] and Holland [13] review some of the recent work on both isothermal and non-isothermal thin-film flows.

In this paper we use the lubrication approximation to investigate the steady flow of slender non-uniform rivulets of a viscous fluid on an inclined plane that is either heated or cooled relative to the surrounding atmosphere. Four non-isothermal situations in which thermocapillary effects play a significant role are considered. The general equations for a slender rivulet subject to gravity, surface tension, thermocapillarity and a constant surface shear stress are derived. Similarity solutions describing a thermocapillary-driven rivulet widening or narrowing due to either gravitational or surface-tension effects on a non-uniformly heated or cooled substrate are obtained. Similarity solutions are also obtained for a gravity-driven and a constant-surface-shear-stress-driven rivulet widening or narrowing due to thermocapillarity on a uniformly heated or cooled substrate. In a companion paper to the present work Holland, Wilson and Duffy [14] study slender dry patches in an infinitely wide film flowing steadily on a heated or cooled inclined plane in the four analogous situations.

2 Problem formulation

We consider the steady flow of a thin symmetric rivulet of viscous fluid with uniform density ρ, viscosity μ, specific heat c and thermal conductivity k_{th} on a heated or cooled plane inclined at an angle α ($0 \leq \alpha \leq \pi$) to the horizontal, when there is an imposed constant shear stress on the free surface and the surface tension of the fluid varies linearly with temperature. We consider both sessile rivulets (when $0 \leq \alpha < \pi/2$) and pendent rivulets (when $\pi/2 < \alpha \leq \pi$) as well as rivulets on a vertical substrate (when $\alpha = \pi/2$). Cartesian coordinates $Oxyz$ with the x axis down the line of greatest slope and the z axis normal to the substrate are adopted, with the substrate at $z = 0$. The edges of the rivulet are at $y = \pm y_e(x)$. The geometry of the problem is shown in Fig. 1. The velocity $\mathbf{u} = (u, v, w) = (u(x, y, z), v(x, y, z), w(x, y, z))$, pressure $p = p(x, y, z)$, and temperature $T = T(x, y, z)$ of the fluid are governed by the mass conservation, Navier–Stokes and energy equations

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$\rho(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g}, \tag{2}$$

$$\rho c(\mathbf{u} \cdot \nabla)T = k_{\mathrm{th}} \nabla^2 T, \tag{3}$$

where $\mathbf{g} = g(\sin\alpha, 0, -\cos\alpha)$ is acceleration due to gravity. On the substrate $z = 0$ the appropriate boundary conditions are zero velocity and a prescribed (in general non-uniform) temperature $T_0 = T_0(x, y)$:

$$\mathbf{u} = \mathbf{0}, \qquad T = T_0; \tag{4}$$

on the free surface $z = h(x, y)$ the appropriate boundary conditions are normal and tangential stress balances and an energy balance, which take the forms

$$\mathbf{n} \cdot \mathbf{T} \cdot \mathbf{n} = -\gamma\kappa, \tag{5}$$

$$\mathbf{t}_1 \cdot \mathbf{T} \cdot \mathbf{n} = \mathbf{t}_1 \cdot \nabla\gamma + \tau, \tag{6}$$

$$\mathbf{t}_2 \cdot \mathbf{T} \cdot \mathbf{n} = \mathbf{t}_2 \cdot \nabla\gamma, \tag{7}$$

$$-k_{\mathrm{th}} \nabla T \cdot \mathbf{n} = \alpha_{\mathrm{th}}(T - T_\infty), \tag{8}$$

together with the kinematic condition on $z = h$, which may be written in the form

$$\bar{u}_x + \bar{v}_y = 0, \tag{9}$$

where

$$\bar{u} = \int_0^h u \, \mathrm{d}z, \quad \bar{v} = \int_0^h v \, \mathrm{d}z \tag{10}$$

are the local fluxes in the longitudinal and transverse (i.e. in the x and y) directions respectively. Here $\mathbf{T}$ denotes the stress tensor of the fluid, $\mathbf{n}$ is the unit normal vector to the free surface, $\mathbf{t}_1$ and $\mathbf{t}_2$ are unit tangential vectors to the free surface in the longitudinal and transverse planes respectively (i.e. in the (x, z) and (y, z) planes, respectively), τ is the imposed constant shear stress on the free surface in the longitudinal plane, T_∞ is the prescribed uniform temperature of the passive atmosphere above the rivulet, $\gamma = \gamma(T)$ is the surface tension, α_{th} is the surface heat-transfer coefficient and $\kappa = \nabla \cdot \mathbf{n}$ is twice the mean curvature of the free surface. The surface tension is assumed to depend linearly on temperature T according to

$$\gamma(T) = \gamma_{\mathrm{r}} - \lambda(T - T_{\mathrm{r}}), \tag{11}$$

where T_{r} is a reference temperature taken to be the temperature of the substrate at some position $x = x_0$, $y = 0$, γ_{r} is the surface tension when $T = T_{\mathrm{r}}$, and $\lambda = -\mathrm{d}\gamma/\mathrm{d}T$ is a positive constant; T_{r} may be greater than or less than T_∞, corresponding to the substrate at $x = x_0$, $y = 0$ being hotter or colder than the atmosphere respectively. The prescribed volume flux of fluid down the rivulet is given by

$$Q = \int_{-y_{\mathrm{e}}}^{y_{\mathrm{e}}} \int_0^h u \, \mathrm{d}z \, \mathrm{d}y, \tag{12}$$

where Q is a positive constant. At the edges of the rivulet $y = \pm y_{\mathrm{e}}$ where $h = 0$ a condition must be specified concerning the contact angle $\beta = \beta(x, y_{\mathrm{e}}(x))$. For example, β may be assumed to satisfy a fixed-contact-angle condition, or to depend on the substrate temperature in a prescribed way; however for our purposes it is not necessary to be specific about this condition.

Following the approach of Smith [2] and Duffy and Moffatt [3] we consider a slender rivulet for which the length scale in the x direction is much greater than that in the y direction which in turn is much greater than that in the z direction, and so we scale the system as follows:

$$
\begin{gathered}
x = lx^*, \quad y = \epsilon l y^*, \quad z = \epsilon\delta l z^*, \quad h = \epsilon\delta l h^*, \\
u = Uu^*, \quad v = \epsilon U v^*, \quad w = \epsilon\delta U w^*, \quad \beta = \delta\beta^*, \\
p = \frac{\mu U}{\delta^2 l} p^*, \quad \tau = \frac{\mu U}{\epsilon\delta l}\tau^*, \quad \gamma = \gamma_{\mathrm{r}}\gamma^*, \quad Q = \epsilon^2\delta l^2 U, \\
T = T_\infty + (T_{\mathrm{r}} - T_\infty)T^*, \quad T_0 = T_\infty + (T_{\mathrm{r}} - T_\infty)T_0^*,
\end{gathered}
\tag{13}
$$

where $\epsilon \ll 1$ and $\delta \ll 1$ are the longitudinal and transverse aspect ratios respectively (i.e. the aspect ratios in the (x, z) and (y, z) planes, respectively), U is the longitudinal velocity scale, and l is the longitudinal length scale. Provided that the appropriate Reynolds and Peclet numbers are sufficiently small (i.e. $R = \epsilon^2\delta^2\rho l U/\mu \ll 1$ and $P = \epsilon^2\delta^2\rho c l U/k_{\mathrm{th}} \ll 1$), the scaled governing equations at leading order are (with superscript stars dropped)

$$u_x + v_y + w_z = 0, \tag{14}$$

$$\epsilon S \sin\alpha + \delta u_{zz} = 0, \tag{15}$$

$$-p_y + v_{zz} = 0, \tag{16}$$

$$-p_z - S\cos\alpha = 0, \tag{17}$$

$$T_{zz} = 0, \tag{18}$$

with the boundary conditions

$$u = v = w = 0, \quad T = T_0 \tag{19}$$

on the substrate $z = 0$, and

$$-p = \frac{1}{C}\gamma h_{yy}, \tag{20}$$

$$u_z = -\frac{1}{\Delta C}(T_x + h_x T_z) + \tau, \tag{21}$$

$$v_z = -\frac{1}{\epsilon^2 \Delta C}(T_y + h_y T_z), \tag{22}$$

$$T_z + BT = 0 \tag{23}$$

on the free surface $z = h$; from (11) the surface tension γ is given by

$$\gamma = 1 - \frac{\delta^2 C}{\epsilon^2 \Delta C}(T - 1), \tag{24}$$

evaluated on $z = h$. The flux condition (12) may now be written as

$$1 = \int_{-y_\mathrm{e}}^{y_\mathrm{e}} \bar{u}\,\mathrm{d}y. \tag{25}$$

At $y = 0$ the free surface satisfies the regularity conditions

$$h_y = 0, \quad h_{yyy} = 0; \tag{26}$$

at the contact line $y = y_\mathrm{e}(x)$ we have

$$h = 0, \quad h_y = -\beta. \tag{27}$$

Four non-dimensional parameters arise in these leading-order equations, namely the Stokes number S, the capillary number C, the thermocapillary number ΔC and the Biot number B, defined by

$$S = \frac{\epsilon \delta^3 \rho g l^2}{\mu U}, \quad C = \frac{\epsilon \mu U}{\delta^3 \gamma_\mathrm{r}}, \quad \Delta C = \frac{\mu U}{\epsilon \delta \lambda (T_\mathrm{r} - T_\infty)}, \quad B = \frac{\epsilon \delta \alpha_\mathrm{th} l}{k_\mathrm{th}}. \tag{28}$$

Integrating (14)–(18) subject to (19) on $z = 0$ and (20)–(23) on $z = h$ yields

$$p = S \cos\alpha (h - z) - \frac{1}{C}\gamma h_{yy}, \tag{29}$$

$$u = \frac{\epsilon S \sin\alpha}{\delta}(2h - z)\frac{z}{2} + \tau z - \frac{1}{\Delta C}\left(\frac{T_0}{1 + Bh}\right)_x z, \tag{30}$$

$$v = -p_y(2h - z)\frac{z}{2} - \frac{1}{\epsilon^2 \Delta C}\left(\frac{T_0}{1 + Bh}\right)_y z, \tag{31}$$

$$w = p_y h_y \frac{z^2}{2} + p_{yy}(3h - z)\frac{z^2}{6} - \frac{\epsilon S \sin\alpha}{\delta} h_x \frac{z^2}{2} + \frac{1}{2\Delta C}\left[\left(\frac{T_0}{1 + Bh}\right)_{xx} + \frac{1}{\epsilon^2}\left(\frac{T_0}{1 + Bh}\right)_{yy}\right] z^2, \tag{32}$$

$$T = T_0\left(1 - \frac{Bz}{1 + Bh}\right), \tag{33}$$

and substituting (30) and (31) into (10) we find

$$\bar{u} = \frac{\epsilon S \sin\alpha}{\delta}\frac{h^3}{3} + \tau\frac{h^2}{2} - \frac{1}{\Delta C}\frac{h^2}{2}\left(\frac{T_0}{1+Bh}\right)_x, \tag{34}$$

$$\bar{v} = -p_y\frac{h^3}{3} - \frac{1}{\epsilon^2\Delta C}\frac{h^2}{2}\left(\frac{T_0}{1+Bh}\right)_y. \tag{35}$$

The kinematic condition (9) now yields the governing partial differential equation for h:

$$\left[\frac{h^3}{3}\left\{S\cos\alpha h - \frac{1}{C}h_{yy} + \frac{\delta^2}{\epsilon^2\Delta C}\left(\frac{T_0}{1+Bh} - 1\right)h_{yy}\right\}_y + \frac{1}{\epsilon^2\Delta C}\frac{h^2}{2}\left(\frac{T_0}{1+Bh}\right)_y\right]_y$$
$$-\left[\frac{\epsilon S\sin\alpha}{\delta}\frac{h^3}{3} + \tau\frac{h^2}{2} - \frac{1}{\Delta C}\frac{h^2}{2}\left(\frac{T_0}{1+Bh}\right)_x\right]_x = 0, \tag{36}$$

which is to be solved subject to (26) and (27), and the flux condition (25), which takes the form

$$1 = \int_{-y_e}^{y_e} \frac{\epsilon S\sin\alpha}{\delta}\frac{h^3}{3} + \tau\frac{h^2}{2} - \frac{1}{\Delta C}\frac{h^2}{2}\left(\frac{T_0}{1+Bh}\right)_x \mathrm{d}y. \tag{37}$$

Equations (36) and (37) are rather general equations for a slender rivulet subject to gravity, surface tension, thermocapillarity and a constant surface shear stress. Particular forms of these equations have been studied previously for both isothermal and non-isothermal flow. Smith [2] obtained a similarity solution describing a non-uniform isothermal gravity-driven rivulet in the absence of surface-tension effects; in this case the two gravity terms in (36) (that is, the terms in S) are dominant with the gravity term dominating the flux condition (37). Duffy and Moffatt [3] obtained the corresponding similarity solution when the dominant balance in (36) is between surface tension (represented by the term in $1/C$) and gravity (represented by the term in $S\sin\alpha$), with gravity again dominating the flux condition (37). Wilson, Duffy and Hunt [5] solved a generalised (but isothermal) version of (36) and (37) to obtain similarity solutions describing a non-uniform rivulet of a non-Newtonian power-law fluid driven by either gravity or a constant surface shear stress. Duffy and Moffatt [9] considered the locally unidirectional gravity-driven flow of an isothermal rivulet, so that the gravity and surface-tension terms in (36) are dominant. Holland, Duffy and Wilson [10] considered the corresponding locally uniform (but not locally unidirectional) non-isothermal flow on a uniformly heated or cooled substrate, in which case the dominant balance in (36) is between gravity, surface tension and thermocapillarity (represented by the term in $1/\epsilon^2\Delta C$).

In the present work we shall obtain similarity solutions in four non-isothermal situations in which thermocapillary effects play a significant role, namely a rivulet driven by thermocapillarity that is widening or narrowing due to either gravity or surface tension, and a rivulet driven by either gravity or a constant surface shear stress that is widening or narrowing due to thermocapillarity.

Following the approach of Smith [2] and Duffy and Moffatt [3] we seek similarity solutions of (36) and (37) for h of the form

$$h = bf(x)G(\eta), \quad \eta = \frac{y}{y_{\mathrm{e}}(x)}, \tag{38}$$

in which the constant b (included for convenience) and the functions $f = f(x)$, $y_{\mathrm{e}} = y_{\mathrm{e}}(x)$ and $G = G(\eta)$ are to be determined, with G satisfying the regularity conditions

$$G'(0) = G'''(0) = 0 \tag{39}$$

and the contact-line condition

$$G(1) = 0, \tag{40}$$

where a prime denotes differentiation with respect to argument. Solutions of the form (38) cannot, in general, satisfy a prescribed contact-angle condition of the type discussed earlier. The isothermal similarity solutions obtained by Smith [2] and Duffy and Moffatt [3] have a similar shortcoming; however, Wilson, Duffy and Davis [6] showed how these solutions can be modified locally near the contact line to accommodate a fixed-contact-angle condition by incorporating sufficiently strong slip at the solid/fluid interface into the model. Similar analyses are presumably possible for the present problems but are not attempted here. Evidently we must have $y_{\mathrm{e}} \geq 0$ and $h \geq 0$ for the solution (38) to be physically relevant, and without loss of generality we therefore take b and f to be positive and y_{e} and G to be non-negative. In the sections that follow, x_0 (which may be infinite) is chosen such that $y_{\mathrm{e}}(x_0) = 0$, and solutions in both $x \leq x_0$ and $x \geq x_0$ will be considered. We denote the height of the rivulet along its centre line by $h_{\mathrm{m}} = h(x, 0)$.

3 Thermocapillary-driven rivulet widening or narrowing due to gravity

In this section we consider a thermocapillary-driven rivulet that is widening or narrowing due to gravity (i.e. the case in which the first and last terms dominate (36) and the thermocapillary term dominates (37)). We consider a rather general non-uniform substrate temperature distribution that depends on x but is independent of y, i.e. $T_0 = T_0(x)$. For later convenience we write T_0 in the form

$$T_0 = 1 - \int_{x_0}^{x} \theta(\tilde{x})\,\mathrm{d}\tilde{x}, \tag{41}$$

satisfying $T_0(x_0) = 1$, $T_{0,x} = -\theta$ and $T_{0,y} = 0$, where $\theta = \theta(x)$ is a prescribed function of x. Setting $S|\cos\alpha| = 1/|\Delta C| = 1$ we find that ϵ, δ and U are given by

$$\epsilon = \left(\frac{\rho g|\cos\alpha|Q\mu}{\lambda^2(T_\mathrm{r} - T_\infty)^2}\right)^{\frac{1}{3}} \ll 1, \quad \delta = \left(\frac{\lambda|T_\mathrm{r} - T_\infty|}{\rho g|\cos\alpha|l^2}\right)^{\frac{1}{2}} \ll 1, \quad U = \left(\frac{\lambda^5|T_\mathrm{r} - T_\infty|^5 Q^2}{\rho g|\cos\alpha|\mu^4 l^6}\right)^{\frac{1}{6}}. \tag{42}$$

The remaining terms in (36) and (37) are negligible provided that

$$\frac{1}{C} \ll 1, \quad B \ll \epsilon^2, \quad \tau \ll 1, \quad |\tan\alpha| \ll \frac{\delta}{\epsilon}, \quad \frac{\delta^2}{\epsilon^2} \ll 1, \tag{43}$$

which, in particular, mean that surface tension, surface heat transfer and surface shear stress must be sufficiently small, and that $\alpha \ll 1$ or $\pi - \alpha \ll 1$ (so that the substrate is horizontal or nearly horizontal). In this case the governing equations (36) and (37) reduce to

$$(h^3 h_y)_y - \frac{3\sigma_T\sigma_\mathrm{c}}{2}(h^2\theta)_x = 0 \tag{44}$$

and

$$1 = \sigma_T\theta \int_{-y_\mathrm{e}}^{y_\mathrm{e}} \frac{h^2}{2}\,\mathrm{d}y \tag{45}$$

at leading order, where we have written

$$\sigma_T = \mathrm{sgn}(T_\mathrm{r} - T_\infty), \quad \sigma_\mathrm{c} = \mathrm{sgn}(\cos\alpha), \tag{46}$$

so that $\sigma_T = \pm 1$ correspond to the substrate at $x = x_0$, $y = 0$ being hotter or colder than the surrounding atmosphere respectively, and $\sigma_\mathrm{c} = \pm 1$ correspond to a sessile or a pendent rivulet respectively. Note that (45) requires $\sigma_T\theta > 0$; in particular, this means that θ must be of one sign.

Seeking a solution of (44) and (45) in the form (38) we have

$$(G^3 G')' + \frac{3\sigma_T \sigma_c C_1}{2b^2}(\eta G^2)' = 0 \tag{47}$$

and

$$1 = \frac{\sigma_T b^2 C_2}{2} \int_{-1}^{1} G^2 \,\mathrm{d}\eta, \tag{48}$$

where

$$C_1 = \frac{\theta y_e y_e'}{f^2} = -\frac{(\theta f^2)' y_e^2}{f^4}, \quad C_2 = \theta f^2 y_e \tag{49}$$

are constants.

In the special case $C_1 = 0$ the equation for G is simply $G' = 0$ with the trivial solution $G = G_0$, where G_0 is an undetermined constant. Evidently this solution cannot satisfy the contact-line condition (40); however we can truncate it at $|\eta| = 1$ to obtain the one-parameter family of solutions (parameterised by the constant $y_e > 0$)

$$h = \left|\frac{1}{\theta y_e}\right|^{\frac{1}{2}} \quad \text{for} \quad |y| \leq y_e, \tag{50}$$

representing a parallel-sided rivulet of arbitrary width whose transverse (but not, in general, longitudinal) profile is uniform.

In the general case $C_1 \neq 0$ the product of C_1 and C_2 in (49) leads to $(y_e^3)' = 3C_1 C_2 \theta^{-2}$. If we define the function $J = J(x)$ by

$$J = \int_{x_0}^{x} \theta(\tilde{x})^{-2} \,\mathrm{d}\tilde{x}, \tag{51}$$

then provided that $C_1 C_2 J \geq 0$ we have

$$y_e = (3C_1 C_2 J)^{\frac{1}{3}}. \tag{52}$$

From C_2 in (49) we find that

$$f = \left(\frac{C_2^2}{3C_1 \theta^3 J}\right)^{\frac{1}{6}}. \tag{53}$$

With the choice $b = |3C_1/2|^{1/2}$ equation (47) reduces to

$$GG' + s\eta = 0, \tag{54}$$

where $s = \sigma_{\mathrm{c}}\sigma_T \mathrm{sgn}(C_1) = \mathrm{sgn}\,(\cos\alpha y_{\mathrm{e}}')$. A real and positive solution for G is possible only when $s = 1$, in which case (54) can be integrated subject to the contact-line condition (40) to yield

$$G = (1-\eta^2)^{\frac{1}{2}} \tag{55}$$

(so that $G(0) = 1$ and $G'(1) = -\infty$). From the flux condition (48) we find that $\sigma_{\mathrm{c}} = C_1 C_2$, where we have made use of the facts that $s = 1$ and

$$\int_{-1}^{1} G^2 \,\mathrm{d}\eta = \int_{-1}^{1} (1-\eta^2)\,\mathrm{d}\eta = \frac{4}{3}. \tag{56}$$

Thus we obtain the unique solution

$$h = \left|\frac{9}{8\theta^3 J}\right|^{\frac{1}{6}} \left(1 - \frac{y^2}{y_{\mathrm{e}}^2}\right)^{\frac{1}{2}}, \quad y_{\mathrm{e}} = |3J|^{\frac{1}{3}}, \tag{57}$$

which is valid provided that $s = \mathrm{sgn}(\cos\alpha(x-x_0)) = 1$. The solution (57) represents both a narrowing ($y_{\mathrm{e}}' < 0$) pendent ($\cos\alpha < 0$) rivulet in $x \le x_0$, and a widening ($y_{\mathrm{e}}' > 0$) sessile ($\cos\alpha > 0$) rivulet in $x \ge x_0$. Provided that $\theta(x_0)$ is finite and non-zero for $|x_0| < \infty$ we have $J = O(x-x_0)$ as $x \to x_0$, and hence $h_{\mathrm{m}} = O(|x-x_0|^{-1/6})$ and $y_{\mathrm{e}} = O(|x-x_0|^{1/3})$ in this limit.

From (42) and (43) the conditions for these solutions to be valid can be expressed as $\epsilon \ll 1$, $\delta \ll 1$,

$$\begin{gathered}\left(\frac{\lambda^4 (T_{\mathrm{r}} - T_\infty)^4 \gamma_{\mathrm{r}}^3}{(\rho g |\cos\alpha|)^5 Q^2 \mu^2}\right)^{\frac{1}{6}} \ll l, \quad 1 \ll \left(\frac{k_{\mathrm{th}}^6 (\rho g |\cos\alpha|)^5 Q^2 \mu^2}{\alpha_{\mathrm{th}}^6 \lambda^7 |T_{\mathrm{r}} - T_\infty|^7}\right)^{\frac{1}{6}}, \quad l \ll \frac{\lambda |T_{\mathrm{r}} - T_\infty|}{\tau}, \\ l \ll \left(\frac{\lambda^7 |T_{\mathrm{r}} - T_\infty|^7}{(\rho g|\cos\alpha|)^5 \tan^6\alpha Q^2 \mu^2}\right)^{\frac{1}{6}}, \quad \left(\frac{\lambda^7 |T_{\mathrm{r}} - T_\infty|^7}{(\rho g |\cos\alpha|)^5 Q^2 \mu^2}\right)^{\frac{1}{6}} \ll l.\end{gathered} \tag{58}$$

We proceed by giving the details of the solution when $C_1 \neq 0$ for a particular choice of the substrate temperature gradient. As a simple example we consider a power-law substrate temperature gradient with $\theta = x^k$, where k is a constant. Considering solutions for $x \ge 0$ (so that $\theta \ge 0$ and $x_0 \ge 0$) we have from (51)

$$J = \begin{cases} \dfrac{x^{1-2k} - x_0^{1-2k}}{1-2k} & k \neq \dfrac{1}{2}, \\[2ex] \log \dfrac{x}{x_0} & k = \dfrac{1}{2}. \end{cases} \tag{59}$$

Evidently a solution with $x_0 = 0$ is possible only for $k < 1/2$, and a solution with $x_0 = \infty$ is possible only for $k > 1/2$. In the general case $k \neq 1/2$ the solution (57) is

$$h = \left| \frac{9(1-2k)}{8x^{3k}(x^{1-2k} - x_0^{1-2k})} \right|^{\frac{1}{6}} \left(1 - \frac{y^2}{y_e^2} \right)^{\frac{1}{2}}, \quad y_e = \left| \frac{3(x^{1-2k} - x_0^{1-2k})}{1-2k} \right|^{\frac{1}{3}}, \tag{60}$$

and in the special case $k = 1/2$ it is

$$h = \left| \frac{9}{8x^{3/2} \log x/x_0} \right|^{\frac{1}{6}} \left(1 - \frac{y^2}{y_e^2} \right)^{\frac{1}{2}}, \quad y_e = |3 \log x/x_0|^{\frac{1}{3}}, \tag{61}$$

both of which are valid provided that $s = 1$. In particular, in the special case of a uniform temperature gradient $\theta = 1$ (that is, when $k = 0$) we find from (60) that $h_m = |9/8(x-x_0)|^{1/6}$ and $y_e = |3(x - x_0)|^{1/3}$.

Figures 2–4 show the solutions for h_m, y_e and h in cases with $k = 0$ ($< 1/2$), $k = 1/2$ and $k = 1$ ($> 1/2$) respectively, for a range of values of x_0. These figures show both narrowing ($y_e' < 0$) pendent ($\cos\alpha < 0$) rivulets in $x \leq x_0$ (represented by the dashed lines in the figures for h_m and y_e), and widening ($y_e' > 0$) sessile ($\cos\alpha > 0$) rivulets in $x \geq x_0$ (represented by the solid lines in the figures for h_m and y_e). As the figures show, for $x_0 > 0$ we have $h_m = O(x^{-k/2})$ and $y_e = O(1)$ as $x \to 0^+$ when $k < 1/2$, and $h_m = O(x^m)$ and $y_e = O(x^n)$ as $x \to 0^+$ when $k > 1/2$, where $m = -(1+k)/6$ and $n = (1-2k)/3$. On the other hand, for $x_0 = 0$ (in which case $k < 1/2$) we have $h_m = O(x^m)$ and $y_e = O(x^n)$ as $x \to 0^+$. Moreover for $x_0 < \infty$ we have $h_m = O(x^m)$ and $y_e = O(x^n)$ as $x \to \infty$ when $k < 1/2$, and $h_m = O(x^{-k/2})$ and $y_e = O(1)$ as $x \to \infty$ when $k > 1/2$. On the other hand, for $x_0 = \infty$ (in which case $k > 1/2$) we have $h_m = O(x^m)$ and $y_e = O(x^n)$ as $x \to \infty$. In the special case $k = 1/2$ for $0 < x_0 < \infty$ we have $h_m = O(|x^{3/2} \log x|^{-1/6})$ and $y_e = O(|\log x|^{1/3})$ both as $x \to 0^+$ and as $x \to \infty$.

Holland [13] gives details of the solution (57) for two further choices of the substrate temperature gradient, namely an exponential temperature gradient and a spatially periodic temperature gradient.

4 Thermocapillary-driven rivulet widening or narrowing due to surface tension

In this section we consider a thermocapillary-driven rivulet that is widening or narrowing due to surface tension (i.e. the case in which the second and last terms dominate (36) and the thermocapillary term dominates (37)). As in section 3, the substrate temperature distribution is taken to be in the form (41). Setting $C = |\Delta C| = 1$ we find that ϵ, δ and U are given by

$$\epsilon = \left(\frac{\gamma_\mathrm{r} Q\mu}{\lambda^2 (T_\mathrm{r} - T_\infty)^2 l^2}\right)^{\frac{1}{5}} \ll 1, \quad \delta = \left(\frac{\lambda |T_\mathrm{r} - T_\infty| Q^2 \mu^2}{\gamma_\mathrm{r}^3 l^4}\right)^{\frac{1}{10}} \ll 1,$$
$$U = \left(\frac{\lambda^7 |T_\mathrm{r} - T_\infty|^7 Q^4}{\gamma_\mathrm{r} \mu^6 l^8}\right)^{\frac{1}{10}}. \tag{62}$$

The remaining terms in (36) and (37) are negligible provided that

$$\frac{\epsilon S \sin\alpha}{\delta} \ll 1, \quad S|\cos\alpha| \ll 1, \quad B \ll \epsilon^2, \quad \tau \ll 1, \quad \frac{\delta^2}{\epsilon^2} \ll 1, \tag{63}$$

which, in particular, mean that gravity, surface heat transfer and surface shear stress must be sufficiently small; however, unlike in the case considered in section 3, there is no restriction on α and so the solutions are valid for $0 \le \alpha \le \pi$. In this case the governing equations (36) and (37) reduce to

$$(h^3 h_{yyy})_y + \frac{3\sigma_T}{2}(h^2\theta)_x = 0 \tag{64}$$

and (45) at leading order, where σ_T is given by (46). As in section 3, θ must be of one sign.

Seeking a solution of (64) and (45) in the form (38) we have

$$(G^3 G''')' + \frac{3\sigma_T C_3}{2b^2}(\eta G^2)' = 0 \tag{65}$$

and (48), where

$$C_3 = \frac{\theta y_\mathrm{e}^3 y_\mathrm{e}'}{f^2} = -\frac{(\theta f^2)' y_\mathrm{e}^4}{f^4} \tag{66}$$

is a constant.

In the special case $C_3 = 0$ the equation for G is simply $G''' = 0$ with the solution $G = G_0(1 - \eta^2)$, where G_0 is an undetermined constant, and hence we obtain the one-parameter family of solutions (parameterised by the constant $y_{\rm e} > 0$)

$$h = \left|\frac{15}{8\theta y_{\rm e}}\right|^{\frac{1}{2}} \left(1 - \frac{y^2}{y_{\rm e}^2}\right), \tag{67}$$

representing a parallel-sided rivulet of arbitrary width whose transverse profile is parabolic.

In the general case $C_3 \neq 0$ the product of C_2 in (49) and C_3 in (66) leads to $(y_{\rm e}^5)' = 5C_2C_3\theta^{-2}$. Provided that $C_2C_3J \geq 0$ (where the function $J = J(x)$ is defined by (51)) we have

$$y_{\rm e} = (5C_2C_3J)^{\frac{1}{5}}, \tag{68}$$

and so from C_2 in (49) we have

$$f = \left(\frac{C_2^4}{5C_3\theta^5 J}\right)^{\frac{1}{10}}. \tag{69}$$

With the choice $b = |3C_3/2|^{1/2}$ equation (65) reduces to

$$GG''' - s\eta = 0, \tag{70}$$

where $s = \sigma_T \mathrm{sgn}(C_3) = \mathrm{sgn}(y_{\rm e}')$. Unlike in the case described in section 3, we cannot find G explicitly in this case, and so equation (70) was solved numerically, using a Runge-Kutta method within the computer algebra package *Mathematica*, subject to (39), (40) and the condition $G(0) = G_0$. Equation (70) was also obtained by Wilson, Duffy and Hunt [5] for a rather different physical problem (namely a constant-surface-shear-stress-driven rivulet of a power-law fluid with strong surface tension) and the present numerical results agree with theirs, although they were calculated using a different method. Figure 5 shows numerically calculated solutions of (70) in the cases (a) $s = 1$ and (b) $s = -1$ for a range of values of G_0. Figure 5 shows that when $s = 1$, G is non-negative everywhere on $[0, 1]$ only if $G_0 \geq G_{0\rm c} \simeq 0.4277$ and always has a single maximum at $\eta = 0$, whereas when $s = -1$, G is non-negative everywhere on $[0, 1]$ for all $G_0 \geq 0$ and has a single maximum at $\eta = 0$ when $G_0 \geq G_0^* \simeq 0.2138$, but otherwise has two equal global maxima $G = G_{\rm m}(G_0)$ at $\eta = \pm\eta_{\rm m}(G_0)$ and a local minimum at $\eta = 0$. A local analysis shows that the behaviour of G as $\eta = 1 - \xi \to 1^-$ is either

$$G = A_0\xi + (A_1 + B_1 \log\xi)\xi^2 + (A_2 + B_2 \log\xi)\xi^3 + o(\xi^3), \tag{71}$$

where the constants B_1, A_2 and B_2 are given by

$$B_1 = -\frac{s}{2A_0}, A_2 = \frac{1}{6A_0^3}\left[sA_0(A_0 + A_1) + \frac{11}{12}\right], B_2 = -\frac{1}{12A_0^3}, \tag{72}$$

but the constants $A_0 > 0$ and A_1 are undetermined locally, or if $s = 1$

$$G = \left(\frac{8}{3}\right)^{\frac{1}{2}} \xi^{\frac{3}{2}} + \left(\frac{1}{6}\right)^{\frac{1}{2}} \xi^{\frac{5}{2}} - \frac{5}{204}\left(\frac{3}{8}\right)^{\frac{1}{2}} \xi^{\frac{7}{2}} + O\left(\xi^{\frac{9}{2}}\right), \tag{73}$$

which has zero slope at $\eta = 1$.

From the flux condition (48) we find that $3I|C_2C_3|/4 = 1$, where we have defined

$$I = \int_{-1}^{1} G^2 \,\mathrm{d}\eta. \tag{74}$$

Figure 6 shows I plotted as a function of G_0 for both $s = 1$ and $s = -1$. As Fig. 6 shows, I satisfies $I \simeq 0.0163$ at $G_0 = 0$ when $s = -1$, $I \simeq 0.1731$ at $G_0 = G_{0\mathrm{c}}$ when $s = 1$, and $I \sim 16G_0^2/15$ as $G_0 \to \infty$.

Thus we obtain the one-parameter family of solutions (parameterised by $G_0 \geq 0$)

$$h = \left|\frac{24}{5I^4\theta^5 J}\right|^{\frac{1}{10}} G\left(\frac{y}{y_\mathrm{e}}\right), \quad y_\mathrm{e} = \left|\frac{20J}{3I}\right|^{\frac{1}{5}}, \tag{75}$$

where G satisfies (70), which is valid for both $s = 1$ and $s = -1$, where $s = \mathrm{sgn}(x - x_0)$. The solution (75) represents both a narrowing ($y_\mathrm{e}' < 0$) rivulet with $s = -1$ in $x \leq x_0$ (which is analogous to the case $\cos\alpha < 0$ in section 3), and a widening ($y_\mathrm{e}' > 0$) rivulet with $s = 1$ in $x \geq x_0$ (which is analogous to the case $\cos\alpha > 0$ in section 3). Provided that $\theta(x_0)$ is finite and non-zero for $|x_0| < \infty$ we have $J = O(x - x_0)$ as $x \to x_0$, and hence $h_\mathrm{m} = O(|x - x_0|^{-1/10})$ and $y_\mathrm{e} = O(|x - x_0|^{1/5})$ in this limit.

From (62) and (63) the conditions for these solutions to be valid can be expressed as $\epsilon \ll 1$, $\delta \ll 1$,

$$\begin{gathered} l \ll \left(\frac{\lambda^{13}|T_\mathrm{r} - T_\infty|^{13}\gamma_\mathrm{r}}{(\rho g \sin\alpha)^{10} Q^4 \mu^4}\right)^{\frac{1}{12}}, \quad l \ll \left(\frac{\lambda^4 (T_\mathrm{r} - T_\infty)^4 \gamma_\mathrm{r}^3}{(\rho g |\cos\alpha|)^5 Q^2 \mu^2}\right)^{\frac{1}{6}}, \\ l \ll \left(\frac{k_\mathrm{th}^2 \gamma_\mathrm{r}}{\alpha_\mathrm{th}^2 \lambda |T_\mathrm{r} - T_\infty|}\right)^{\frac{1}{2}}, \quad l \ll \frac{\lambda|T_\mathrm{r} - T_\infty|}{\tau}, \quad \frac{\lambda|T_\mathrm{r} - T_\infty|}{\gamma_\mathrm{r}} \ll 1. \end{gathered} \tag{76}$$

As an example we consider again the case of a power-law substrate temperature gradient with $\theta = x^k$ for $x \geq 0$. In the general case $k \neq 1/2$ the solution (75) is

$$h = \left| \frac{24(1-2k)}{5I^4 x^{5k}(x^{1-2k} - x_0^{1-2k})} \right|^{\frac{1}{10}} G\left(\frac{y}{y_e}\right), \quad y_e = \left| \frac{20(x^{1-2k} - x_0^{1-2k})}{3(1-2k)I} \right|^{\frac{1}{5}}, \tag{77}$$

and in the special case $k = 1/2$ it is

$$h = \left| \frac{24}{5I^4 x^{5/2} \log x/x_0} \right|^{\frac{1}{10}} G\left(\frac{y}{y_e}\right), \quad y_e = \left| \frac{20 \log x/x_0}{3I} \right|^{\frac{1}{5}}. \tag{78}$$

In particular, in the special case of a uniform temperature gradient $\theta = 1$ (that is, when $k = 0$) we find from (77) that $h_m = |24/5I^4(x - x_0)|^{1/10}$ and $y_e = |20(x - x_0)/3I|^{1/5}$.

In the case $s = 1$ the solutions for h_m, y_e and h are qualitatively similar to those given in section 3 and so are omitted for the sake of brevity. However in the case $s = -1$ the transverse profile of the rivulet has one global maximum at $y = 0$ (as in section 3) for $G_0 \geq G_0^*$ but two equal global maxima and a local minimum at $y = 0$ for $G_0 < G_0^*$. This latter behaviour is illustrated in Fig. 7, which shows the solution for h in the case $\theta = x$ (i.e. $k = 1$) when $s = -1$, $G_0 = 0.1$ ($< G_0^*$) and $x_0 = 10$.

For $x_0 > 0$ we have $h_m = O(x^{-k/2})$ and $y_e = O(1)$ as $x \to 0^+$ when $k < 1/2$, and $h_m = O(x^m)$ and $y_e = O(x^n)$ as $x \to 0^+$ when $k > 1/2$, where $m = -(1 + 3k)/10$ and $n = (1-2k)/5$. On the other hand, for $x_0 = 0$ (in which case $k < 1/2$) we have $h_m = O(x^m)$ and $y_e = O(x^n)$ as $x \to 0^+$. Moreover for $x_0 < \infty$ we have $h_m = O(x^m)$ and $y_e = O(x^n)$ as $x \to \infty$ for $k < 1/2$, and $h_m = O(x^{-k/2})$ and $y_e = O(1)$ as $x \to \infty$ for $k > 1/2$. On the other hand, for $x_0 = \infty$ (in which case $k > 1/2$) we have $h_m = O(x^m)$ and $y_e = O(x^n)$ as $x \to \infty$. In the special case $k = 1/2$ for $0 < x_0 < \infty$ we have $h_m = O(|x^{5/2} \log x|^{-1/10})$ and $y_e = O(|\log x|^{1/5})$ both as $x \to 0^+$ and as $x \to \infty$.

Holland (2002) gives details of the solution (75) for the two further choices of the substrate temperature gradient mentioned in section 3.

5 Gravity-driven rivulet widening or narrowing due to thermocapillarity

In this section we consider a gravity-driven rivulet on a uniformly heated or cooled substrate (so that $T_0 \equiv 1$) that is widening or narrowing due to thermocapillarity (i.e. the case in which the fourth and fifth terms dominate (36) and the gravity term dominates (37)). Setting $\epsilon S \sin\alpha/\delta = B/\epsilon^2|\Delta C| = 1$ we find that ϵ, δ and U are given by

$$\epsilon = \left(\frac{\alpha_{\mathrm{th}}\lambda|T_{\mathrm{r}} - T_\infty|}{k_{\mathrm{th}}\rho g \sin\alpha l}\right)^{\frac{1}{2}} \ll 1, \quad \delta = \left(\frac{k_{\mathrm{th}}^2 \rho g \sin\alpha Q\mu}{\alpha_{\mathrm{th}}^2 \lambda^2 (T_{\mathrm{r}} - T_\infty)^2 l^2}\right)^{\frac{1}{3}} \ll 1,$$
$$U = \left(\frac{k_{\mathrm{th}}(\rho g \sin\alpha)^2 Q^2}{\alpha_{\mathrm{th}}\lambda|T_{\mathrm{r}} - T_\infty|\mu l}\right)^{\frac{1}{3}}. \tag{79}$$

The remaining terms in (36) and (37) are negligible provided that

$$\frac{1}{C} \ll 1, \quad \tau \ll 1, \quad \frac{\delta}{\epsilon} \ll |\tan\alpha|, \tag{80}$$

which, in particular, mean that surface tension and surface shear stress must be sufficiently small, and that α is not near 0 or π (so that the substrate is not horizontal or nearly horizontal). Similarity solutions cannot be found when $B \neq 0$, and therefore we restrict our attention to the adiabatic limit $B \to 0$. In this case the governing equations (36) and (37) reduce to

$$\sigma_T (h^3)_{yy} + 2(h^3)_x = 0 \tag{81}$$

and

$$1 = \int_{-y_{\mathrm{e}}}^{y_{\mathrm{e}}} \frac{h^3}{3}\,\mathrm{d}y \tag{82}$$

at leading order, where σ_T is given by (46).

Seeking a solution of (81) and (82) in the form (38) we have

$$(G^3)'' + 6\sigma_T\left(\frac{f' y_{\mathrm{e}}^2}{f} G^3 - y_{\mathrm{e}} y_{\mathrm{e}}' \eta G^2 G'\right) = 0 \tag{83}$$

and

$$1 = \frac{b^3 f^3 y_{\mathrm{e}}}{3} \int_{-1}^{1} G^3\,\mathrm{d}\eta. \tag{84}$$

In this case we find that the relevant forms for f and y_{e} are

$$f = (cx)^{-\frac{1}{6}}, \quad y_{\mathrm{e}} = (cx)^{\frac{1}{2}}, \tag{85}$$

with the constant c to be determined; without loss of generality we have taken $x_0 = 0$ here. Then from (83) the equation for G is

$$G' - \frac{c\sigma_T}{3}\eta G = 0, \tag{86}$$

which can be integrated to yield

$$G = G_0 \exp\left(\frac{c\sigma_T}{6}\eta^2\right), \tag{87}$$

where $G_0 = G(0)$. The solution (87) does not satisfy the contact-line condition (40), but if this condition is dropped then we can interpret the solution when $c\sigma_T < 0$ (which satisfies $G \to 0$ as $\eta \to \infty$) as an infinitely wide "rivulet". (Note that the solution when $c\sigma_T > 0$ is also infinitely wide, but satisfies $G \to \infty$ as $\eta \to \infty$.) With the choice $b = 3^{1/3}$, and with G substituted from (87), equation (84) is replaced by

$$1 = \int_{-\infty}^{+\infty} G^3 \,\mathrm{d}\eta = G_0^3 \int_{-\infty}^{+\infty} \exp\left(\frac{c\sigma_T}{2}\eta^2\right) \mathrm{d}\eta = G_0^3 \left(-\frac{2\pi\sigma_T}{c}\right)^{\frac{1}{2}}, \tag{88}$$

and hence

$$c = -2\pi\sigma_T G_0^6. \tag{89}$$

Thus we obtain the unique solution

$$h = \left(-\frac{9\sigma_T}{2\pi x}\right)^{\frac{1}{6}} \exp\left(\frac{\sigma_T y^2}{6x}\right), \tag{90}$$

which is valid provided that $\sigma_T x < 0$. Hence the unique solution (90) represents both a "narrowing" rivulet ($y_{\mathrm{e}}' < 0$) in $x < 0$ when the substrate is heated ($\sigma_T = 1$), and a "widening" rivulet ($y_{\mathrm{e}}' > 0$) in $x > 0$ when the substrate is cooled ($\sigma_T = -1$). The free-surface temperature $(1 + Bh)^{-1} = 1 - Bh + O(B^2)$ is a decreasing function of h. In the case when the substrate is heated ($\sigma_T = 1$) the surface tension $\gamma = 1 + \sigma_T\delta^2 Ch(1 + Bh)^{-1} = 1 + \sigma_T\delta^2 Ch + O(B)$ is an increasing function of h, and hence there is a gradient of surface tension that drives a transverse flux inwards away from $y = \pm y_{\mathrm{e}}$ and towards $y = 0$, causing the rivulet to narrow and deepen. In the case when the substrate is cooled ($\sigma_T = -1$) the converse holds, causing the rivulet to widen and shallow. Figure 8 shows the solution for h in the case $\sigma_T = -1$; the corresponding solution in the case $\sigma_T = 1$ is a reflection of this in the plane $x = 0$.

From (79) and (80) the conditions for these solutions to be valid can be expressed as $\epsilon \ll 1$, $\delta \ll 1$,

$$\left(\frac{k_{\mathrm{th}}^{13}(\rho g\sin\alpha)^5\gamma_{\mathrm{r}}^6Q^2\mu^2}{\alpha_{\mathrm{th}}^{13}\lambda^{13}|T_{\mathrm{r}}-T_\infty|^{13}}\right)^{\frac{1}{7}} \ll l, \quad l \ll \frac{k_{\mathrm{th}}(\rho g\sin\alpha)^5Q^2\mu^2}{\alpha_{\mathrm{th}}\lambda|T_{\mathrm{r}}-T_\infty|\tau^6}, \quad \frac{k_{\mathrm{th}}^7(\rho g\sin\alpha)^5Q^2\mu^2}{\alpha_{\mathrm{th}}^7\lambda^7|T_{\mathrm{r}}-T_\infty|^7\tan^6\alpha} \ll l. \tag{91}$$

6 Shear-stress-driven rivulet widening or narrowing due to thermocapillarity

In this section we consider a shear-stress-driven rivulet on a uniformly heated or cooled substrate (so that, as in section 5, $T_0 \equiv 1$) that is widening or narrowing due to thermocapillarity (i.e. the case in which the fourth and sixth terms dominate (36) and the shear-stress term dominates (37)). Setting $\tau = B/\epsilon^2|\Delta C| = 1$ we find that ϵ, δ and U are given by

$$\begin{gathered}\epsilon = \left(\frac{\alpha_{\mathrm{th}}^2\lambda^2(T_{\mathrm{r}}-T_\infty)^2Q\mu}{k_{\mathrm{th}}^2l^3\tau^3}\right)^{\frac{1}{5}} \ll 1, \quad \delta = \left(\frac{k_{\mathrm{th}}^3Q\mu\tau^2}{\alpha_{\mathrm{th}}^3\lambda^3|T_{\mathrm{r}}-T_\infty|^3l^3}\right)^{\frac{1}{5}} \ll 1,\\ U = \left(\frac{k_{\mathrm{th}}Q^2\tau^4}{\alpha_{\mathrm{th}}\lambda|T_{\mathrm{r}}-T_\infty|\mu^3l}\right)^{\frac{1}{5}}.\end{gathered} \tag{92}$$

The remaining terms in (36) and (37) are negligible provided that

$$\frac{\epsilon S\sin\alpha}{\delta} \ll 1, \quad S|\cos\alpha| \ll 1, \quad \frac{1}{C} \ll 1, \tag{93}$$

which, in particular, mean that gravity and surface tension must be sufficiently small; however, unlike in the case considered in section 5, there is no restriction on α and so the solutions are valid for $0 \le \alpha \le \pi$. As in section 5, we restrict our attention to the adiabatic limit $B \to 0$. In this case the governing equations (36) and (37) reduce to

$$\sigma_T(h^3)_{yy} + 3(h^2)_x = 0 \tag{94}$$

and

$$1 = \int_{-y_{\mathrm{e}}}^{y_{\mathrm{e}}} \frac{h^2}{2}\,\mathrm{d}y \tag{95}$$

at leading order, where σ_T is given by (46).

Seeking a solution of (94) and (95) in the form (38) we have

$$(G^3)'' + \frac{6\sigma_T}{b}\left(\frac{f' y_{\mathrm{e}}^2}{f^2} G^2 - \frac{y_{\mathrm{e}} y_{\mathrm{e}}'}{f}\eta G G'\right) = 0 \tag{96}$$

and

$$1 = \frac{b^2 f^2 y_{\mathrm{e}}}{2} \int_{-1}^{1} G^2 \,\mathrm{d}y. \tag{97}$$

In this case we find that the relevant forms for f and y_{e} are

$$f = (cx)^{-\frac{1}{5}}, \quad y_{\mathrm{e}} = (cx)^{\frac{2}{5}}, \tag{98}$$

with the constant c to be determined; without loss of generality we have again taken $x_0 = 0$ here. With the choice $b = |2c/5|$ equation (96) reduces to

$$G' - s\eta = 0, \tag{99}$$

where $s = \sigma_T \mathrm{sgn}(c)$. A real and positive solution for G is possible only when $s = -1$, in which case (99) can be integrated subject to (40) to yield

$$G = \frac{1}{2}(1 - \eta^2) \tag{100}$$

(so that $G(0) = 1/2$ and $G'(1) = -1$). From (97) we have

$$c = -\sigma_T \left(\frac{375}{8}\right)^{\frac{1}{2}}. \tag{101}$$

Thus we obtain the unique solution

$$h = \frac{1}{2}\left(-\frac{45\sigma_T}{2x}\right)^{\frac{1}{5}} \left(1 - \frac{y^2}{y_{\mathrm{e}}^2}\right), \quad y_{\mathrm{e}} = \left(\frac{375x^2}{8}\right)^{\frac{1}{5}}, \tag{102}$$

which is valid only for $\sigma_T x < 0$. Hence the unique solution (102) represents both a narrowing rivulet ($y_{\mathrm{e}}' < 0$) in $x < 0$ when the substrate is heated ($\sigma_T = 1$), and a widening rivulet ($y_{\mathrm{e}}' > 0$) in $x > 0$ when the substrate is cooled ($\sigma_T = -1$). This behaviour is qualitatively the same as that described in section 5 and has the same physical explanation. Figure 9 shows the solution for h in the case $\sigma_T = -1$; the corresponding solution in the case $\sigma_T = 1$ is a reflection of this in the plane $x = 0$.

From (92) and (93) the conditions for these solutions to be valid can be expressed as $\epsilon \ll 1$, $\delta \ll 1$,

$$\frac{k_{\mathrm{th}}(\rho g \sin\alpha)^5 Q^2 \mu^2}{\alpha_{\mathrm{th}} \lambda |T_{\mathrm{r}} - T_\infty| \tau^6} \ll l, \quad \frac{k_{\mathrm{th}}^6 (\rho g |\cos\alpha|)^5 Q^2 \mu^2}{\alpha_{\mathrm{th}}^6 \lambda^6 (T_{\mathrm{r}} - T_\infty)^6 \tau} \ll l, \quad \frac{k_{\mathrm{th}}^2 \gamma_{\mathrm{r}} \tau}{\alpha_{\mathrm{th}}^2 \lambda^2 (T_{\mathrm{r}} - T_\infty)^2} \ll l. \tag{103}$$

7 Conclusions

In this paper we used the lubrication approximation to investigate the steady flow of slender non-uniform rivulets of a viscous fluid on an inclined plane that is either heated or cooled relative to the surrounding atmosphere. Four non-isothermal situations in which thermocapillary effects play a significant role were considered. We derived the general equations for a slender rivulet subject to gravity, surface tension, thermocapillarity and a constant surface shear stress. Similarity solutions describing a thermocapillary-driven rivulet widening or narrowing due to either gravitational or surface-tension effects on a non-uniformly heated or cooled substrate were obtained, and we presented examples of these solutions when the substrate temperature gradient depends on the longitudinal coordinate x according to a general power law. In the case of strong gravitational effects we found a unique solution representing both a narrowing pendent rivulet and a widening sessile rivulet whose transverse profile always has a single global maximum at $y = 0$. In the case of strong surface-tension effects we found a one-parameter family of solutions (parameterised by $G_0 \geq 0$) representing both a narrowing and a widening rivulet. In this case we found that for a widening rivulet a solution is possible only for $G_0 \geq G_{0c} \simeq 0.4277$ and the transverse profile of the rivulet always has a single global maximum at $y = 0$, whereas for a narrowing rivulet there is a solution for all G_0 whose transverse profile has a single global maximum at $y = 0$ for $G_0 \geq G_0^* \simeq 0.2138$, but otherwise has two equal global maxima and a local minimum at $y = 0$. We also obtained unique similarity solutions whose transverse profiles always have a single maximum describing both a gravity-driven and a constant-surface-shear-stress-driven rivulet widening or narrowing due to thermocapillarity on a uniformly heated or cooled substrate. The solutions in both cases represent both a narrowing rivulet on a heated substrate and a widening rivulet on a cooled substrate (albeit with infinite width in the gravity-driven case). In the Appendix we show that the equations governing the free-surface profile in sections 3, 5 and 6 (but *not* section 4) may be converted (by appropriate changes of variables) to a nonlinear diffusion problem of standard type for which there are known similarity solutions.

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Appendix

In this appendix we show that the problems defined by (44) and (45) in section 3, (81) and (82) in section 5, and (94) and (95) in section 6 (but *not* (64) and (45) in section 4) may be converted to a nonlinear diffusion problem of standard type of the form

$$u_t = (u^n u_x)_x, \quad q = \int_{-a(t)}^{a(t)} u \, \mathrm{d}x \qquad (n \geq 0) \tag{A1}$$

for $u = u(x,t)$ in the unbounded domain $-\infty < x < \infty$ for $t \geq 0$, to be solved subject to some appropriate initial condition (at $t = 0$), and hence in these cases we recover the solutions found earlier from known similarity solutions of (A1). In (A1) q is a prescribed constant, and $x = \pm a(t)$ are the edges of the diffusing zone, outside of which $u(x,t) = 0$; in the case $n = 0$ the edges are at $x = \pm\infty$ for $t > 0$.

For $n > 0$ problem (A1) has the similarity solution [15, 16][1]

$$u(x,t) = \left(\frac{nq^2}{2(n+2)K^2 t}\right)^{\frac{1}{n+2}} (1-\eta^2)^{\frac{1}{n}}, \quad \eta = \frac{x}{a}, \quad a = \left(\frac{2(n+2)q^n t}{nK^n}\right)^{\frac{1}{n+2}} \quad \text{if } |\eta| \leq 1 \tag{A2}$$

for $t > 0$ (with $u = 0$ for $|\eta| > 1$), where

$$K = \int_{-1}^{1} (1-\eta^2)^{\frac{1}{n}} \, \mathrm{d}\eta = \sqrt{\pi}\frac{\Gamma(\frac{1}{n}+1)}{\Gamma(\frac{1}{n}+\frac{3}{2})}. \tag{A3}$$

For $n = 0$ problem (A1) has the similarity solution

$$u(x,t) = \frac{q}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right) \tag{A4}$$

[1] Note that there is a typographical error in equation (32) of [15]: the coefficient of x^2 should be squared; however, their expression (37) for the position of the advancing front is correct.

for $-\infty < x < \infty$ and $t > 0$; more generally for $n = 0$ problem (A1) has the solution

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} u(\xi,0) \exp\left(-\frac{(x-\xi)^2}{4t}\right) \mathrm{d}\xi \tag{A5}$$

for $t > 0$. The solution (A4) may be obtained from (A5) with the choice $u(x,0) = q\delta(x)$, where $\delta(x)$ denotes the Dirac delta function.

In section 3 we found that the thickness h of a slender thermocapillary-driven rivulet that is widening or narrowing due to gravity satisfies (44) and (45). Substituting

$$h(x,y) = \left(\frac{3\sigma_T u(J,y)}{\theta}\right)^{\frac{1}{2}}, \quad J = \int_{x_0}^{x} \frac{1}{\theta(\tilde{x})^2} \,\mathrm{d}\tilde{x}, \quad y = Y \tag{A6}$$

with $u \geq 0$ we obtain

$$(uu_Y)_Y = \sigma_{\mathrm{c}} u_J, \quad \frac{2}{3} = \int_{-Y_{\mathrm{e}}}^{Y_{\mathrm{e}}} u \,\mathrm{d}Y, \tag{A7}$$

where $Y_{\mathrm{e}}(J) = y_{\mathrm{e}}(x)$. This is of the form (A1) with $n = 1$ and $q = 2/3$, and so from (A2) and (A3) we find that $K = 4/3$, and that with x and t identified with Y and $\sigma_{\mathrm{c}} J$ (> 0) respectively, a solution for u is

$$u = \frac{1}{2(3\sigma_{\mathrm{c}} J)^{\frac{1}{3}}} \left(1 - \frac{Y^2}{Y_{\mathrm{e}}^2}\right), \quad Y_{\mathrm{e}} = (3\sigma_{\mathrm{c}} J)^{\frac{1}{3}}; \tag{A8}$$

thus a solution for h is given by (57) for $J \cos\alpha > 0$.

In section 5 we found that the thickness h of a slender gravity-driven rivulet that is widening or narrowing due to thermocapillarity satisfies (81) and (82). Substituting

$$h(x,y) = (3u)^{\frac{1}{3}}, \quad x = -2\sigma_T X, \quad y = Y \tag{A9}$$

with $u \geq 0$ we obtain

$$u_{YY} = u_X, \quad 1 = \int_{-\infty}^{\infty} u \,\mathrm{d}Y. \tag{A10}$$

This is of the form (A1) with $n = 0$ and $q = 1$, and so from (A4) we find that a solution for u is

$$u = \frac{1}{\sqrt{4\pi X}} \exp\left(-\frac{Y^2}{4X}\right) \tag{A11}$$

in $X > 0$; thus a solution for h is given by (90) for $\sigma_T x < 0$. Moreover from (A5) the solution for h satisfying $h(0,y) = h_0(y)$ is

$$h = \left(-\frac{1}{2\pi\sigma_T x}\right)^{\frac{1}{6}} \left[\int_{-\infty}^{\infty} h_0(\xi)^3 \exp\left(\frac{(y-\xi)^2}{2\sigma_T x}\right) \mathrm{d}\xi\right]^{\frac{1}{3}} \tag{A12}$$

for $\sigma_T x < 0$. The similarity solution (90) may be obtained from (A12) with the choice $h_0(y) = [3\delta(y)]^{1/3}$.

In section 6 we found that the thickness h of a slender shear-stress-driven rivulet that is widening or narrowing due to thermocapillarity satisfies (94) and (95). Substituting

$$h(x, y) = u^{\frac{1}{2}}, \quad x = -2\sigma_T X, \quad y = Y \tag{A13}$$

with $u \geq 0$ we obtain

$$(u^{\frac{1}{2}} u_Y)_Y = u_X, \quad 2 = \int_{-Y_\mathrm{e}}^{Y_\mathrm{e}} u \, \mathrm{d}Y, \tag{A14}$$

where $Y_\mathrm{e}(X) = y_\mathrm{e}(x)$. This is of the form (A1) with $n = 1/2$ and $q = 2$, and so from (A2) and (A3) we find that $K = 16/15$ and that a solution for u is

$$u = \left(\frac{45}{128X}\right)^{\frac{2}{5}} \left(1 - \frac{Y^2}{Y_\mathrm{e}^2}\right)^2, \quad Y_\mathrm{e} = \left(\frac{375X^2}{2}\right)^{\frac{1}{5}}; \tag{A15}$$

thus a solution for h is given by (102) for $\sigma_T x < 0$.

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Figure captions

FIGURE 1. Geometry of the problem.

FIGURE 2. A thermocapillary-driven rivulet widening or narrowing due to gravity: solutions for (a) h_{m} and (b) y_{e} given by (57) in the case $\theta = 1$ for both narrowing ($y_{\mathrm{e}}' < 0$) pendent ($\cos\alpha < 0$) rivulets in $x \leq x_0$ (represented by the dashed lines), and widening ($y_{\mathrm{e}}' > 0$) sessile ($\cos\alpha > 0$) rivulets in $x \geq x_0$ (represented by the solid lines) when $x_0 = 0, 1, ..., 10$, together with three-dimensional plots of h given by (57) in (c) the sessile case when $x_0 = 1$, and (d) the pendent case when $x_0 = 10$.

FIGURE 3. As for Fig. 2 except that $\theta = x^{1/2}$ and (a) and (b) are exclusive of $x_0 = 0$.

FIGURE 4. As for Fig. 2 except that $\theta = x$ and (a) and (b) are exclusive of $x_0 = 0$.

FIGURE 5. Numerically calculated rivulet profiles $G = G(\eta)$ obtained from (70) plotted as a function of η when (a) $s = 1$ and (b) $s = -1$ for a range of values of G_0.

FIGURE 6. The integral $I = I(G_0)$ given by (74) plotted as a function of G_0 for both $s = 1$ and $s = -1$.

FIGURE 7. A thermocapillary-driven rivulet widening or narrowing due to surface tension: numerically calculated three-dimensional plot of h given by (77) in the case $\theta = x$ when $s = -1$, $G_0 = 0.1$ and $x_0 = 10$.

FIGURE 8. A gravity-driven rivulet widening or narrowing due to thermocapillarity: three-dimensional plot of h given by (90) in the case $\sigma_T = -1$.

FIGURE 9. A shear-stress-driven rivulet widening or narrowing due to thermocapillarity: three-dimensional plot of h given by (102) in the case $\sigma_T = -1$.