Tools or crutches? Apparatus as a sense-making aid in mathematics teaching with children with moderate learning difficulties.

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Introduction

There have been shifts in the perceived value of apparatus over the last 20 years with discussion about the disparity between the practice and purpose of practical materials within arithmetic teaching (Threlfall,1996). Practical apparatus is seen as an important artifact in classroom practice for supporting learning and teaching in mathematics with evidence that children develop mathematical understanding through interacting with objects (Gray, Pitta & Tall, 2000; Steffe, von Glaserfeld, Richards & Cobb, 1983; Piaget, 1965). It would seem judicious that all children be afforded the opportunity to build understanding by working with materials however there is a danger that concrete materials come to be used mechanistically by pupils without commensurate understanding (Moyer, 2001; Clements, 1999; Clements & McMillen, 1996; Threlfall, 1996, Cobb, 1995). This paper challenges a view of concrete materials as artifacts used within a rigid instructional sequence that particular children are perceived to require or not, as the case may be. It contends that it is more useful to consider the function of these materials as *tools*, artifacts used flexibly and selectively by pupils to make sense of mathematics rather than as *crutches*, devices which may support procedural competency in mathematics but with no guarantees of understanding.

The specific use of apparatus to show pupils how to carry out mathematical procedures is set out in instructional texts (Thyer & Maggs, 1992) and it is a teaching approach recommended to support pupils with learning difficulties (Westwood, 1993). Yet giving pupils specific tactics to employ in their solution to problems is arguably undesirable and unnecessary; it can hamper pupils with learning difficulties in learning with understanding (Behrend, 2003; Baroody, 1989) and there is evidence that this group of learners are capable of inventing their own solution strategies (Baroody, 1996; Behrend, 1994). The ways in which pupils with moderate learning difficulties are able to use apparatus to make sense of problems challenges the current orthodoxy of using materials to explicate procedures.

Distinctions have been made between procedural and conceptual understanding in mathematics. Conceptual knowledge relates to 'knowing why' and involves an understanding of the network of mathematical relationships. Procedural knowledge involves 'knowing how to' and consists of knowing specific sequences of procedures to be carried out (Hiebert & Lefevre, 1986). This distinction relates closely to what Skemp (1976) described as 'relational' and 'instrumental' understanding. In his seminal paper Skemp recognised the importance of both types of understanding distinguishing between them using a geographical analogy. Instrumental understanding involves having a number of fixed and independent maps, whereas relational understanding involves having an integrated mental map.

The significance of the relationship between procedural and conceptual understanding and the extent to which teachers can and should foster this connection has been emphasised (Askew et al., 1997; Gray & Tall, 1993). In this respect there are two important considerations regarding the use of concrete materials in mathematics teaching. The first is that the materials themselves carry no actual mathematical information (Hiebert et al., 1997), the second issue relates to pedagogy and how materials come to be used in instruction. In the absence of clear and informed knowledge and understanding of the potential role of concrete materials as tools that can be used autonomously by pupils to build conceptual understanding there is a danger that teachers will persist in maintaining a default position where materials are used to demonstrate procedures for pupils to reenact. This position may be sustained by traditions common to educational support. Traditional responses to supporting pupils with learning difficulties within special education settings have been informed by diagnostic and remedial approaches (Thomas & Loxley, 2007). These approaches have resulted in teachers being distracted from what it is children actually do in their learning and instead foster an over-reliance on questionable and sometimes prescriptive pedagogies that are seen to be in some way unique and relevant to children with learning difficulties (Thomas & Loxley, ibid).

Pupils with moderate learning difficulties persist with primitive strategy use in solving arithmetical problems (Geary, Hamson & Hoard, 2000; Jordan & Montani, 1997; Ostad, 1999, 1997) at the expense of development in their mathematical thinking (Dowker, 2004; Baroody, 2003). The consequences of concrete materials being used as crutches are evidenced in some studies. Ostad's studies (ibid.) found that pupils with learning difficulties were reliant on concrete materials rather than mental strategies to solve mathematical problems; furthermore they did not discard these materials and move onto using mental strategies in solving arithmetical problems. The extent to which this reluctance to abandon concrete materials as an issue of instruction requires consideration. Over-reliance on concrete materials as artifacts for generating correct answers is problematic as it can restrict children from progressing onto more efficient strategies that come about through growth in mathematical thinking (Carpenter & Moser, 1982).

Constructivism

Constructivist theory views knowledge as actively constructed by the learner. The idea that mathematics learning should be a sense-making process has been convincingly argued (Twomey-Fosnot & Dolk, 2001; Anghileri, 2000; Fennema & Romberg, 1999; Hiebert et al., 1997) with constructivist theory underpinning these arguments. Research into classroom practice (Watson, 1996) has demonstrated the efficacy of constructivist approaches with pupils with moderate learning difficulties and Watson has called for the development of constructivist practices across the curriculum (Watson, 2001).

From a constructivist perspective effective learning involves children constructing mathematical relationships for themselves (Twomey-Fosnot & Dolk, 2001, Carpenter et al., 1999; Askew et al., 1997, Hiebert et al., op.cit.). The use of concrete materials to build mathematical meaning is consistent with a constructivist philosophy when this apparatus is used by pupils to make sense of problems (Carpenter et al., 1999). The extent to which all pupils, including those with moderate learning difficulties, are afforded opportunities to use materials in ways that supports construction of these relationships can be linked to teachers' knowledge and beliefs not only about learners (Yackel &

Rasmussen, 2003; Franke & Kazemi, 2001; Carpenter et al., 1989) but also about pedagogy (Carpenter et al., 1988; Shulman, 1986).

In their report on the effective teaching of numeracy, Askew et al. (op.cit.) identified the most effective teachers of numeracy as ‘connectionists’. Connectionist teachers demonstrate a sense-making approach to mathematics learning that is rooted in constructivism. They do not view learning mathematics as simply being about the assimilation and recall of number facts, rather they consciously encourage pupils to develop their understanding of the relationships within the number system and to establish connections between concepts and processes. This model of teaching takes into account the difference between the mathematical understanding of teachers’ and that of pupils (Bills, 1998) and is characterised by a culture of learning evidence by focused discussion between the pupils themselves and between the pupils and the teacher.

The challenge

There is a diverse range of materials and visual aids available in mathematics instruction in classrooms; these include concrete materials or manipulatives such as unifix cubes, multilink blocks, Dienes material or base 10 materials, numberlines and hundred squares. The challenge for class teachers is one of how to structure lessons so that pupils with learning difficulties engage in mathematical activities that encourage them to make sense of the mathematics they are learning while at the same time attempting to develop basic skills and meet the needs of performance expectations (Bottge et al., 2007). Children may require concrete materials to make sense of problems initially, but it is insufficient to use manipulatives as crutches solely for the purpose of carrying out a procedure; to make sense of problems children need to reflect on their actions (Clements, 1999). Empirical studies have shown that children will use manipulatives without prior formal instruction as tools to make sense of word problems by modelling out the language of the problem (Carpenter, Fennema, Franke, Levi & Empson, 1999; Carpenter & Moser, 1982). These quite different uses of tools will be explored drawing from classroom observations of pupils with moderate learning difficulties.

Data

The data being used for the analysis in this article were collected as part of a much larger doctoral study. This paper is not reporting on the study per se but is using observational data to recount what children with moderate learning difficulties actually did with the materials in response to problems designed by their teachers. There is also discussion of interview data where teachers describe the function of concrete materials in their teaching.

Classroom observations and interviews

Observations took place in three primary special schools in Scotland for pupils with moderate learning difficulties. Cognitively Guided Instruction (CGI) (Carpenter et al., 1999) provided a pedagogical framework that was used as a professional development programme with the participating teachers. CGI has been developed in Madison, Wisconsin over the last twenty five or so years and is a research-based pedagogy involving the use of word problems in mathematics instruction. It is founded on research into children's mathematical thinking and how this thinking is reflected in children's solutions to problems posed.

Twelve teachers took part in development sessions which amounted to a full in-service day. Within this allocated time teachers were introduced to different types of mathematical word problems and how children's solution strategies related to these problem types. Within the context of group activities involving the application of CGI each teacher recorded the engagement of pupils within their class. Teachers had complete autonomy in the design and management of these CGI sessions. These teacher-accounts were supported by researcher observations of classroom practice. Prior to applying CGI in practice the teachers were interviewed about the use of concrete materials with pupils with moderate learning difficulties.

Examples of word problems and how pupils responded

The problematic nature of concrete materials as artifacts for developing procedural competency has been stated, yet concrete materials can also be used by pupils as tools to actively build understanding of the mathematics which they are exploring (Hiebert et al., 1997). The following examples from classroom observations demonstrate how pupils with moderate learning difficulties used manipulatives and in some examples graphic representation in this way to solve word problems as outlined by Carpenter et al. (1999).

The Charlie Bucket Problem

29-(4 x 5) = y

The above equation would certainly be a challenging and perhaps unlikely problem to present to a pupil with learning difficulties with no prior experience of formal multiplication problems let alone compound arithmetical problems. Yet when it was presented in this context, 'Charlie Bucket had 29 sweets. He gave the other 4 children 5 sweets each. How many sweets did he have left?', it was solved elegantly.

Figure 1: Direct modelling a solution to the Charlie Bucket problem

Through engaging with the language of the problem and using materials to enact the story the pupil arrived at a solution. The teacher's annotated account and photographic evidence (Figure 1) shows that the pupil counted out 29 little people, he made four sets of five and then counted the remaining 9 items. Following the language of the problem in this way materials were used to make sense of the problem. The learner not only determined the correct answer but was also beginning to engage at a counting level (Anghileri, 2000), in mathematical concepts to which he had not been formally introduced, in this case multiplication.

The Verruca Salt Problem

A Primary 6 teacher gave the following problem:

Verruca Salt comes from a very rich family. She has 93 dolls! 48 of her dolls have blonde hair. How many do not?

A traditional approach to solving this problem might involve the pupils being shown to set out materials to represent the amounts (Thyer & Maggs, op.cit.). This could be done using base ten materials and a tens and units chart. The pupils would then set out 9 tens and 3 ones. They would then be shown to move one of the tens into the ones column exchanging it for 10 individual ones. Pupils would then be taught to remove 8 from this bundle of 13 and remove 4 from the bundle of 8 tens, resulting in the correct answer.

Leaving aside the problem of whether a pupil might actually be able to hold onto that particular sequence of steps in order to reproduce correct answers consistently, there is a more fundamental issue concerning conceptual understanding, notably, will the pupil have understood the reason for moving one of the tens across and exchanging for 10 ones? Working with concrete materials in this prescribed way becomes a constant re-enactment of explicitly taught procedures with the hope that this will lead to an ability to be able to work more abstractly with numbers. In this traditional approach which focuses on procedural skills, once children have mastered the manipulation of concrete materials on a board they are then shown how to represent this sequence of manipulation by setting out what has been referred to as the 'dance of the symbols' (Davis, 1996) (Figure 2).

$$\begin{array}{r} {}^{8}\cancel{9}{}^{1}3 \\ -\ \ 4\ \ 8 \end{array}$$

Figure 2: The ‘dance of the symbols’

This method of instruction is systematic, it can be broken down into small–steps and can facilitate the ability to carry out a particular procedure. It reflects instructional approaches that are driven by behaviourist models exemplified by a lock-step sequence that facilitates target setting - pupils work in subtraction within 10, then beyond ten, without decomposition and then with decomposition. Rather than being presented with conceptually more challenging problems, pupils are simply presented with larger numbers, single digits progress to tens and units which in turn leads to working with hundreds tens and units.

A further issue with the example given above is that in explicitly showing children how it can be solved by a process of subtraction, it may be overlooked that some children may solve this problem by counting on from 48. Indeed some children might even use materials to keep track of their counting.

Figure 3 shows the attempt by Malcolm, a ten-year old pupil with moderate learning difficulties, at solving the Verruca Salt problem. Initially Malcolm recognised that the problem could be solved by subtraction and he responded by setting out the standard algorithm as he had been previously instructed. This taught procedure was his default starting position, it can be seen in the upper left hand quadrant of figure 3. However once he had set out the symbols Malcolm was unable to recall what he was to do next. He then set about direct modelling a solution to the problem but having no strategy that would allow him to keep track of his counting he became aware of the inefficiency of his method and gave up. In this problem Malcolm evidently had not understood a taught process and fell back to an inefficient and finally ineffective counting strategy. This example highlights the importance of connecting the procedural and the conceptual and the role of the teacher in providing appropriate learning opportunities that foster these connections and the development of more efficient calculation strategies.

Figure 3: Malcolm's attempted solution to the Verucca Salt problem (93 - 48)

Using knowledge of children's solution strategies to inform teaching

In discussion the class teacher expressed concern that Malcolm had forgotten a previously taught procedure. However, in response to Malcolm's effort the teacher learned that he needed to be able to work with groups of ten and count in tens. The teacher then developed problems for Malcolm that encouraged working with sets of ten.

A simpler separating problem was given to some of the class, 'There are 27 children on the bus. 6 of them get off. How many children are still on the bus?', Figure 4 shows how one pupil solved the problem. There is a striking similarity in how this pupil and Malcolm directly modelled the problem. They both worked with sets of ten and in Malcolm's solution in Figure 5 we can see that he has marked his sets of ten and also included the number sentence that represents the problem. Malcolm's solution with his sets of ten marked off, demonstrates his potential to move from this extended graphic representation to more efficient and abstract counting strategies. The experiences of using materials to model out problems encouraged him to connect his conceptual and procedural understanding and fostered the solution of more difficult subtraction problems with understanding rather than simply attempting to recall a taught procedure.

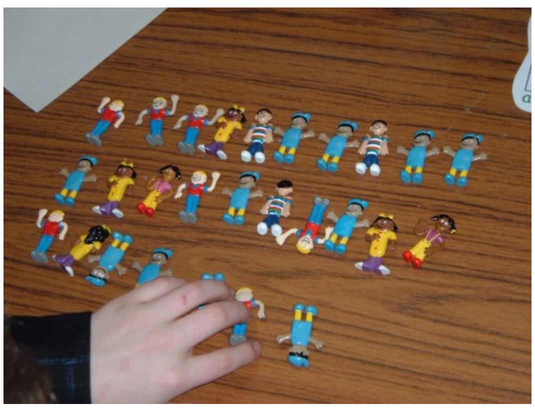

Figure 4:Pupil directly modelling 27-6 problem

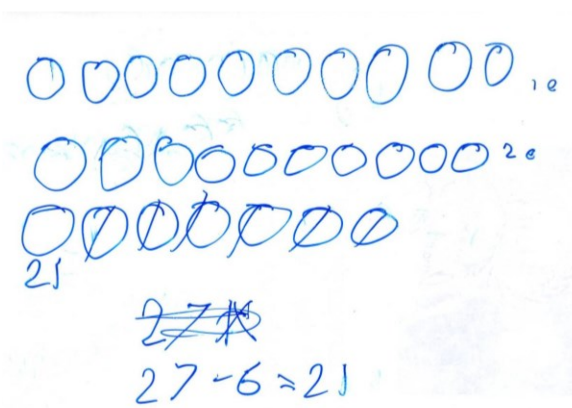

Figure 5: Malcolm's solution to the bus problem

There was also evidence that pupils were recognising the purpose of materials as sense-making tools and were learning from each other. When presented with the problem: 'St Mirren play Motherwell in the cup. The game lasted 90 minutes. St Mirren had the ball for 56 minutes. How long did Motherwell have it?' Malcolm encouraged another pupil, Pat, who had made an error in his mental calculation, to model a solution. The following commentary was annotated by the teacher,

> Pat: It's 33. I started at 90 and counted back in tens to 50. Then I counted back 3 units to 33.
>
> Malcolm: No think Pat, it's 90 minutes, I did it with the tens (base 10 material). I took away 56, it's 34.
>
> Pat checked using materials and agreed.

This episode demonstrates the significant difference in how both pupils conceptualised the function of concrete materials. Pat had been reluctant to use materials as he perceived them as 'babyish', a support or crutch which he no longer required. Malcolm on the other hand was using the apparatus as a tool to make sense of problems; he was not dependent on concrete materials and used them selectively depending on the nature of the problem. Further evidence gathered by the teacher showed that Malcolm's encouragement moved Pat towards a more flexible use of concrete materials.

Teachers' accounts of the use of apparatus

Interviews with classroom teachers revealed that they considered the use of concrete materials to be an important resource in mathematics instruction with pupils with moderate learning difficulties but there was no evidence to show that the function of these materials had been considered. In other words, the teachers did not give any accounts of materials being used by pupils in ways that helped them to construct understanding of mathematical problems, they emphasised the importance of concrete materials as artifacts for practising rehearsed procedures rather than for investigating and determining solutions.

‘concrete is used to practice’

‘children would use them [concrete materials] to get the answer to the sum they are doing’

‘they are used as a kind of crutch to begin with to give them confidence’

‘I will also make them do that [teaching commutativity] with concrete materials’

Teachers also described materials as artifacts that could be used to demonstrate a particular procedure for pupils to replicate. Several teachers described laying out base ten materials to show children how to carry out specific procedures for addition and subtraction.

‘children are gathered round and me showing one or two examples myself and then each of the children having an opportunity to experience the materials and we go through examples together’

‘if you can illustrate [an operation] along with concrete materials, that will support their learning’

‘using practical materials and I would do it by first showing them how to do it’

‘you would use them for demonstrating and they would practice that’

‘[I] have to show them physically how to use the concrete materials’

Concrete materials were described by most of the teachers as part of a progression of aids. A variety of artifacts were accounted for as computational aids for example: concrete materials, numberlines, hundred squares and multiplication grids. Several teachers aimed to move children from one aid and onto a different one; how pupils actually used these aids was not part of the discourse. The use of particular aids was considered to represent a ‘strategy’, rather than the strategy being described in terms of the process used by the pupil. So for example in accounts of the use of numberlines for addition and subtraction correct answers would be arrived at by counting on or back the correct number of steps. Hundred squares were used for teaching ‘adding on ten’; there was evidence of pupils being shown that the answer could be found by ‘dropping’ a line, similarly subtraction of ten would mean reading the line above. Multiplication problems could be solved by matching the corresponding numbers on the axes of the grid. Each of these procedures, if executed correctly, would result in correct answers, but pupils would not necessarily understand the underlying mathematical concepts being taught. The following accounts demonstrate this view of apparatus as part of an instructional sequence with autonomy resting with the teacher.

‘It would generally be sorting material to start with, then going onto blocks and then numberlines’

‘if they come in using cubes I try to move them onto the number line’

‘you have got to introduce them to other strategies, you would use a numberline’

‘who decides? I decide, I am working with them, I know what level they are at’ ‘we have obviously counting materials, cubes, dinosaurs, teddy bears and various things like that which I think have good mileage when it comes to addition and subtraction’

Tools or Crutches

The evidence from the classroom observations and teacher interviews showed that teachers and pupils used concrete materials in different ways. The pupils demonstrated that they were able to use materials in a sense-making way, for example although Malcolm was getting stuck in the Verruca Salt problem he was still trying to make sense of it. Discussion with the teachers however showed that they had not considered the use of materials in this way instead they used concrete materials to demonstrate procedures for pupils to practice. Arguably to contain the use of materials to this latter function is restrictive and constrains pupils with moderate learning difficulties from using materials in more flexible ways.

A distinction being made between tools and crutches concerns choice and dependency. In a constructivist classroom pupils determine their own solution strategies and teachers employ pedagogies that support flexible responses from pupils (Carpenter et al., 1999). Restriction of this choice can result in pupils’ perception that there is a single correct procedure that has been explicated by the teacher which needs to be executed in order to arrive at a correct solution. In this respect materials can be considered to be useful classroom artifacts in three different ways, as *sense-making tools,* as *demonstrational tools* and as *computational tools*. From a constructivist perspective there is a fundamental difference between materials being used by pupils as *sense-making tools* and teachers using materials as *demonstrational tools* to expound a procedure which consequently leads to students using materials as *computational tools* with which they re-enact the taught sequence.

It is this latter function of concrete materials that is being challenged; while such an approach may promote procedural competency, commensurate growth in conceptual understanding is not guaranteed (Threlfall, 1996; Baroody, 1989). The possibility exists that pupils with learning difficulties will hang onto materials as crutches to allow them to carry out procedures and so remain hampered from moving onto more abstract strategies. Concerns about which computational aids pupils should be using, for example, cubes, number lines or hundred squares are misplaced; it is more useful for teachers to think about the way in which learners' mathematical thinking is developed by using particular tools (Hiebert et al, 1997).

In the knowledge that children's arithmetical thinking is different from adults' it makes sense not to impose on children procedures that are based on more sophisticated knowledge. For example many adults might solve the following problem by subtraction:
Tony has 4 football stickers in his collection. How many more will he need to get to have 9 in his collection?

Although this problem can indeed be solved by subtraction there is nothing in the language of the problem that suggests a separating action to a child, in fact it is a problem of joining: $4 + x = 9$. Developmentally, this is a pivotal type of problem for children. A child who uses concrete materials to model a solution needs to be able to plan ahead. Firstly the child needs to make a set of 4, then he or she counts on to 9, the new items being counted must be kept separate from the first set. Children who are at an emergent stage of direct modelling will not be able to solve this problem if they are unable to maintain these two distinct sets. It is futile to go beyond this and show children how to solve this type of problem by other procedures such as subtraction. This type of procedural instruction may promote the 'how to' but the 'why' is likely to remain unanswered. Thus by providing pupils with procedures which they can replicate through the use of an assortment of apparatus, the apparatus becomes a crutch which is used mechanistically to reach an answer and which pupils become reluctant to discard. This mechanistic use of materials is very different from a flexible use of materials as tools for building understanding.

Children who struggle in learning mathematics will not only hang onto materials as a crutch but will also use inefficient counting procedures that restrict them from moving onto more effective solution strategies (Gray & Tall, op.cit.). However children's intuitive counting strategies are often closely linked to direct modeling and it becomes an issue of effective dynamic assessment to utilise this knowledge of children's solution strategies to inform teaching (Carpenter et al., 1999). Using concrete materials to act out and make sense of problems affords children the opportunity to move towards more abstract thinking as these external actions come to be internalised (Gray, Pitta & Tall, 2000). From a constructivist perspective learning is seen as a generative process; as children's understanding of number and awareness of the complexity and relationships between numbers grows they become able to solve problems in a wider variety of ways.

Conclusion

Several years ago Watson (2001) indicated the potential of constructivist approaches in working with children with moderate learning difficulties. Within the domain of mathematics the development of a pedagogy underpinned by constructivist theory has shown that the participating pupils with moderate learning difficulties were able to use materials in ways that fostered meaningful learning, However this required teachers to be aware of the purpose of these materials and for many teachers this meant a reconsideration of the function of concrete materials within their classrooms. The challenge now is one of developing teachers' knowledge of children's mathematical thinking so that they can effectively establish appropriate classroom conditions for all learners. In the absence of this kind of knowledge there is the possibility that teachers will maintain support structures and propose the use of materials in such a way that these function as crutches; at best this may aid procedural competency. There is a need to consider the kind of learning conditions that have to be in place to allow children to learn with understanding in mathematics; the use of materials as sense-making tools is central to this process.

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