On input-to-state stability of stochastic retarded systems with Markovian switching

Lirong Huang and Xuerong Mao

Abstract

This note develops a Razumikhin-type theorem on pth moment input-to-state stability of hybrid stochastic retarded systems (also known as stochastic retarded systems with Markovian switching), which is an improvement of an existing result. An application to hybrid stochastic delay systems verifies the effectiveness of the improved result.

Index Terms

stochastic systems, time delay, Razumikhin-type theorems, ISS, Markov chain.

I. Introduction

Since Markov jump linear systems were firstly introduced in early 1960s (see, e.g., [26], [33] and [42]), the hybrid systems driven by continuous-time Markov chains have been widely employed to model many practical systems where they may experience abrupt changes in system structure and parameters such as failure prone manufacturing, power systems, solar-powered systems and battle management in command, control and communication systems (see [1], [6], [21], [26], [34] and references therein). An area of particular interest has been the stability analysis of this class of hybrid systems and its applications to automatic control (see, e.g., [4], [10], [26] and [33]). When time delays and environmental noise are taken into account, which are often encounterd in real systems and may be the cause of poor performance and

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instability, the hybrid systems are described with stochastic functional differential equations with Markovian switching and called hybrid stochastic retarded systems (HSRSs). One of the most important HSRSs that frequently appear in engineering is those called hybrid stochastic delay systems (HSDSs), which are also known as stochastic delay systems with Markovian switching (SDSwMS) and described with stochastic differential delay equations with Markovian switching (see, e.g., [21], [23], [24] and [41]).

Recently, hybrid stochastic retarded systems (HSRSs) have been widely used since stochastic modelling plays an important role in many branches of science and engineering. Consequently, stability analysis of HSRSs and HSDSs has been studied by many works, see, e.g., [9], [17], [19], [21], [39], [41] and [42]. Among the key results, Mao et al. (see [17], [23], [24]) and Huang et al. ([9]) proposed the Razumikhin-type theorems on stability of hybrid stochastic retarded systems and their applications to hybrid stochastic delay systems. The Razumikhin method is developed to cope with the difficulty arisen from the large, fast varying and nondifferentiable time delays (see, e.g., [21] and [23]). This note is to improve the the Razumikhin-type theorem proposed in [9] and make it more applicable (see Remark 3.2 and Example 4.1).

II. Notation

Throughout the note, unless otherwise specified, we shall employ the following notation. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets) and $\mathbb{E}[\cdot]$ be the expectation operator with respect to the probability measure. Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. If x, y are real numbers, then $x \vee y$ denotes the maximum of x and y, and $x \wedge y$ stands for the minimum of x and y. Let $|\cdot|$ denote the Euclidean norm in R^n. Let $\tau \geq 0$ and $C([-\tau, 0]; R^n)$ denote the family of all continuous R^n-valued functions φ on $[-\tau, 0]$ with the norm $\|\varphi\| = \sup\{|\varphi(\theta)| : -\tau \leq \theta \leq 0\}$. Let $C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$ be the family of all $\mathcal{F}_0$-measurable bounded $C([-\tau, 0]; R^n)$-valued random variables $\xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$. For $p > 0$ and $t \geq 0$, denote by $L^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$ the family of all $\mathcal{F}_t$-measurable $C([-\tau, 0]; R^n)$-valued random processes $\phi = \{\phi(\theta) : -\tau \leq \theta \leq 0\}$ such that $\sup_{-\tau\leq\theta\leq 0}\mathbb{E}|\phi(\theta)|^p < \infty$. We let $\mathcal{K}$ denote the class of continuous strictly increasing functions μ from R_+ to R_+ with $\mu(0) = 0$. Let $\mathcal{K}_\infty$ denote the class of functions $\mu \in \mathcal{K}$ with

$\mu(r) \to \infty$ as $r \to \infty$. Functions in $\mathcal{K}$ and $\mathcal{K}_\infty$ are called class $\mathcal{K}$ and $\mathcal{K}_\infty$ functions, respectively. If $\mu \in \mathcal{K}$, its inverse function is denoted by μ^{-1} with domain $[0, \mu(\infty))$. We denote by $\mu \in V\mathcal{K}$ and $\mu \in C\mathcal{K}$ if $\mu \in \mathcal{K}$ and μ is convex and concave, respectively. In this note, a function $\beta : R_+ \times R_+ \to R_+$ is said to be of class $\mathcal{KL}$ if for each fixed t the mapping $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed s $\beta(s, t)$ is decreasing to zero on t as $t \to \infty$. We also let $\mathcal{L}_\infty^l$ denote the class of essentially bounded functions $u : R_+ \to R^l$ with $||u||_\infty = \operatorname{ess\,sup}_{t \geq 0} |u(t)| < \infty$.

Let $r(t), t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$P\{r(t+\Delta) = j : r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & if\ i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & if\ i = j, \end{cases}$$

where $\Delta > 0$ and $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$. Assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. It is known that almost all sample paths of $r(t)$ are right-continuous step functions with a finite number of simple jumps in any finite subinterval of $R_+ := [0, \infty)$.

Let us consider an n-dimensional HSRS

$$\mathrm{d}x(t) = f(x_t, t, r(t), u_d(t))\mathrm{d}t + g(x_t, t, r(t), u_d(t))\mathrm{d}B(t) \tag{1}$$

on $t \geq 0$ with initial data $x_0 = \{x(\theta) : -\tau \leq \theta \leq 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$ and $r(0) = r_0 \in S$, where $x_t = \{x(t+\theta) : -\tau \leq \theta \leq 0\}$ is regarded as a $C([-\tau, 0]; R^n)$-valued random variable and $u_d \in \mathcal{L}_\infty^l$ the disturbance input. Moreover, $f : C([-\tau, 0]; R^n) \times R_+ \times S \times R^l \to R^n$ and $g : C([-\tau, 0]; R^n) \times R_+ \times S \times R^l \to R^{n \times m}$ are measurable functions with $f(0, t, i, 0) \equiv 0$ and $g(0, t, i, 0) \equiv 0$ for all $t \geq 0$. So equation (1) admits a trivial solution $x(t; 0) \equiv 0$. We assume that f and g are sufficiently smooth so that equation (1) has a unique solution on $t \geq -\tau$ (see, e.g., [12], [15], [16], [17], [18], [21], [22], [25], [29] and [41]), which is denoted by $x(t; x_0, r(0))$ or $x(t; \xi, r_0)$ in this note. It should be noted that equation (1) is a very general type of equation and includes stochastic differential equations, stochastic delay differential equations, integro-differential equations and those with Markovian switching. Much more equations are also included in equation (1) (see, e.g., [7]).

Let $C^{2,1}(R^n \times R_+ \times S; R_+)$ denote the family of all nonnegative functions $V(x, t, i)$ on $R^n \times R_+ \times S$ that are twice continuously differentiable in x and once in t. If $V \in C^{2,1}(R^n \times$

$R_+ \times S; R_+)$, define an operator associated with system (1), $\mathcal{L}$, from $C([-\tau,0];R^n) \times R_+ \times S$ to R by

$$\begin{aligned} \mathcal{L}V(x_t,t,i) &= V_t(x,t,i) + V_x(x,t,i)f(x_t,t,i,u_d) \\ &+ \frac{1}{2}trace\left[g^T(x_t,t,i,u_d)V_{xx}(x,t,i)g(x_t,t,i,u_d)\right] + \sum_{j=1}^{N}\gamma_{ij}V(x,t,j), \end{aligned} \tag{2}$$

where $V_t(x,t,i) = \frac{\partial V(x,t,i)}{\partial t}$, $V_x(x,t,i) = \left(\frac{\partial V(x,t,i)}{\partial x_1}, \cdots, \frac{\partial V(x,t,i)}{\partial x_n}\right)$ and $V_{xx}(x,t,i) = \left(\frac{\partial^2 V(x,t,i)}{\partial x_i \partial x_j}\right)_{n\times n}$.

The purpose of this note is to develop the Razumihkin-type theorem on pth moment input-to-state stability (ISS) of HSRSs and its applications. For definitions of pth moment stability and input-to-state stability, readers are referred to, e.g., [8], [9], [11], [13], [28], [31], [32] and [35]. Let us introduce the definition of pth moment ISS of HSRSs, which is consistent with the definition of ISS for deterministic systems (see, e.g., [11], [31], [32] and [35]).

Definition 2.1: The system (1) is said to be pth ($p > 0$) moment input-to-state stable (ISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that the solution $x(t) = x(t;\xi,r_0)$ satisfies

$$\mathbb{E}|x(t)|^p \le \beta(\mathbb{E}||\xi||^p, t) + \gamma(||u_d||_\infty) \quad \forall t \ge 0 \tag{3}$$

for any essentially bounded input $u_d \in \mathcal{L}_\infty^l$ and any initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];R^n)$, $r_0 \in S$.

III. Razumikhin-type theorem on ISS of HSRSs

As the main result of this note, we present a Razumikhin-type theorem on pth moment ISS of HSRSs (1) as follows.

Theorem 3.1: Let $p > 0$, $u \in V\mathcal{K}_\infty$, $v \in \mathcal{K}_\infty$ and $\lambda \in \mathcal{K}$. Assume that there exists a function $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$ such that

$$u(|x|^p) \le V(x,t,i) \le v(|x|^p), \quad \forall (x,t,i) \in R^n \times [-\tau,\infty) \times S \tag{4}$$

and, moreover, for all $1 \le i \le N$,

$$\mathbb{E}\mathcal{L}V(\phi,t,i) \le \lambda(|u_d(t)|) - \mathbb{E}w(\phi(0),i) \tag{5}$$

for all $t \ge 0$ and those $\phi \in L^p_{\mathcal{F}_t}([-\tau,0];R^n)$ satisfying

$$\min_{k\in S} \mathbb{E}V(\phi(\theta), t+\theta, k) < \mathbb{E}q(\phi(0),t,i) \tag{6}$$

on $-\tau \le \theta \le 0$, where $w : R^n \times S \to R_+$ is a nonnegative function such that there is $\bar{w} \in \mathcal{K}_\infty$ with $w(x,i) \ge \bar{w}(|x|)$ and $\lim_{|x|\to\infty} \frac{\bar{w}(|x|)}{v(|x|^p)} > 0$ for all $i \in S$; $q : R^n \times R_+ \times S \to R$ is a function such that $q(x,t,i) - V(x,t,i) \ge \zeta(|x|)$ for all $(x,t,i) \in R^n \times [-\tau,\infty) \times S$ with $\zeta \in \mathcal{K}_\infty$ and $\lim_{|x|\to\infty} \frac{\zeta(|x|)}{v(|x|^p)} > 0$. Then system (1) is pth moment ISS.

In order to prove this theorem, let us present the following useful lemmas

Lemma 3.1: Let $V(t) = V(x(t),t,r(t))$ for $t \ge 0$, then $\mathbb{E}V(t)$ is continuous on $t \ge 0$.

Proof For any initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];R^n)$, write $x(t) = x(t;\xi)$ and extend $r(t)$ to $[-\tau,0)$ by setting $r(t) = r(0) = r_0$ for all $t \in [-\tau,0)$. For convenience of the readers, the generalized Itô's formula is cited as follows (see [30] and [41])

$$
\begin{aligned}
V(x(t),t,r(t)) &= V(x(0),0,r(0)) + \int_0^t \mathcal{L}V(x_s,s,r(s))\mathrm{d}s \\
&\quad + \int_0^t V_x(x(s),s,r(s))g(x_s,s,r(s))\mathrm{d}B(s) \\
&\quad + \int_0^t \int_R [V(x(s),s,r(0)+h(r(s),l)) - V(x(s),s,r(s))]\mu(\mathrm{d}s,\mathrm{d}l) \quad (7)
\end{aligned}
$$

for all $t \ge 0$, where function $h(\cdot,\cdot)$ and martingale measure $\mu(\cdot,\cdot)$ are defined as, e.g., (2.6) and (2.7) in [41] (see also [6] and [2]).

Since $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];R^n)$, we can find an integer k_0 such that $||\xi|| < k_0$ $a.s.$. For any integer $k > k_0$, define the stopping time

$$\rho_k = \inf\{t \ge 0 : |x(t)| \ge k\}\,, \tag{8}$$

where we set $\inf\emptyset = \infty$ as usual. Note that $x(t)$ is continuous and so are $|x(t)|$ and $v(|x(t)|)$ on $t \ge -\tau$. Clearly, $\rho_k \to \infty$ almost surely as $k \to \infty$. Moreover, since $x_0 = \xi \in C^b_{\mathcal{F}_0}([-\tau,0];R^n)$, $\mathbb{E}V(x(0),0,r(0)) \le \mathbb{E}v(|\xi(0)|) \le v(k_0)$. It then follows from (7) that

$$\mathbb{E}V(x(t_k),t_k,r(t_k)) = \mathbb{E}V(x(0),0,r(0)) + \mathbb{E}\int_0^{t_k} \mathcal{L}V(x_s,s,r(s))\mathrm{d}s \tag{9}$$

where $t_k = t \wedge \rho_k$. So, letting $k \to \infty$, by Fubini's theorem, we have

$$\mathbb{E}V(t) = \mathbb{E}V(0) + \mathbb{E}\int_0^t \mathcal{L}V(x_s,s,r(s))\mathrm{d}s = \mathbb{E}V(0) + \int_0^t \mathbb{E}\mathcal{L}V(x_s,s,r(s))\mathrm{d}s \tag{10}$$

for all $t \ge 0$. This implies $\mathbb{E}V(t)$ is continuous on $t \ge 0$.

Lemma 3.2: For any $t \ge 0$, there is $a_w > 0$ such that $\mathbb{E}w(x,i) \ge a_w$ for all $i \in S$ whenever $\mathbb{E}V(x,t,i) \ge a_v > 0$.

Proof It immediately follows the desired conclusion if we show there is $\mu_w \in \mathcal{K}_\infty$ such that

$$\mathbb{E}\bar{w}(|x(t)|) \geq \mu_w(a_v) \tag{11}$$

whenever $\mathbb{E}v(|x|^p) \geq \mathbb{E}V(x,t,i) \geq a_v > 0$.

Fix t for the moment. We define a nondecreasing function $b : R_+ \to R_+$ as

$$b(y) = \inf_{|x|^p \geq v^{-1}(y/2)} \frac{\bar{w}(|x|)}{v(|x|^p)}, \quad y \geq 0. \tag{12}$$

By property of function $\bar{w}(\cdot)$, $b(y) > 0$ when $y > 0$. So, for any $a_v > 0$, we have

$$\mathbb{E}\bar{w}(|x|) \geq \int_{|x|^p \geq v^{-1}(\frac{a_v}{2})} \bar{w}(|x|)\mathrm{d}\mathbb{P} \geq b(a_v) \int_{v(|x|^p) \geq \frac{a_v}{2}} v(|x|^p)\mathrm{d}\mathbb{P} \geq \frac{a_v b(a_v)}{2}$$

whenever $\mathbb{E}v(|x|^p) \geq \mathbb{E}V(x,t,i) \geq a_v$. Inequality (11) holds with $\mu_w(a_v) = \frac{a_v b(a_v)}{2}$.

Lemma 3.3: For any $t \geq 0$, there is $a_q > 0$ such that $\mathbb{E}q(x,t,i) \geq a_q + \mathbb{E}V(x,t,i)$ for all $i \in S$ whenever $\mathbb{E}V(x,t,i) \geq a_v > 0$.

Proof It is noted that $\mathbb{E}q(x,t,i) - \mathbb{E}V(x,t,i) \geq \mathbb{E}\zeta(|x|)$ for all $t \geq 0$. According to the property of function$\zeta(|x|)$, the rest of the proof is similar to that of Lemma 3.2 and hence omitted.

We can now begin to prove Theorem 3.1.

Proof Denote $\alpha_\lambda = \lambda(||u_d||_\infty)$ and $\bar{V}_0 = u(\mathbb{E}||\xi||^p)$. Without loss of generality, assume $0 < \mu_w^{-1}(2\alpha_\lambda) < u(\sup_{-\tau \leq \theta \leq 0} \mathbb{E}|\xi(\theta)|^p) \leq \bar{V}_0$. For any $t \geq 0$, by Lemma 3.2, $\mathbb{E}w(x(t),i) \geq 2\alpha_\lambda$ whenever $\mathbb{E}V(x,t,i) \geq \mu_w^{-1}(2\alpha_\lambda)$ for all $i \in S$. By Lemma 3.3, there is $a > 0$ such that $\mathbb{E}q(x,t,i) - \mathbb{E}V(x,t,i) \geq a$, $i \in S$, whenever $\mathbb{E}V(x,t,i) \geq \mu_w^{-1}(2\alpha_\lambda)$. Let J be the minimal nonnegative integer such that $M_0 = \mu_w^{-1}(2\alpha_\lambda) + Ja > \bar{V}_0$. Moreover, let $\tilde{\tau} = \tau \vee \frac{M_0}{\alpha_\lambda}$ and $t_j = j\tilde{\tau}$ for $j = 0, 1, 2, \cdots, J$. We claim that

$$\mathbb{E}V(x(t), t, r(t)) \leq \bar{V}_0 \wedge M_j \tag{13}$$

for all $t \geq t_j$, where $M_j = \mu_w^{-1}(2\alpha_\lambda) + (J-j)a$ and $j = 0, 1, 2, \cdots, J$.

First we show that

$$\mathbb{E}V(x(t), t, r(t)) \leq \bar{V}_0, \quad \forall\, t \geq t_0\,. \tag{14}$$

Suppose that $t_a = \inf\{t > t_0 : \mathbb{E}V(x(t), t, r(t)) > \bar{V}_0\} < \infty$. Since $\mathbb{E}V(x(t), t, r(t))$ is continuous on $t \geq 0$, there exist a pair of constants t_b and t_c such that $t_0 \leq t_b \leq t_a < t_c$

and

$$\begin{cases} \mathbb{E}V(x(t),t,r(t)) = \bar{V}_0, & t = t_b; \\ \bar{V}_0 < \mathbb{E}V(x(t),t,r(t)) < \bar{V}_0 + a, & t_b < t \le t_c. \end{cases} \tag{15}$$

However, by equation (10) and condition (5), we have

$$\mathbb{E}V(x(t),t,r(t)) = \mathbb{E}V(x(t_b),t_b,r(t_b)) + \int_{t_b}^{t} \mathbb{E}\mathcal{L}V(x_s,s,r(s))\mathrm{d}s \le \bar{V}_0 - \alpha_\lambda(t-t_b) < \bar{V}_0$$

for every $t \in (t_b, t_c]$, which contradicts (16). So inequality (14) must be true.

We further show that $\mathbb{E}V(x(t),t,r(t)) \le M_1$ for all $t \ge t_1$. Let $\tau_1 = \inf\{t \ge t_0 : \mathbb{E}V(x(t),t,r(t)) \le M_1\}$. If $\tau_1 > t_1$, then, $\forall t_0 \le t \le t_1$, we have

$$\begin{aligned} \mathbb{E}q(x(t),t,r(t)) &\ge \mathbb{E}V(x(t),t,r(t)) + a > M_1 + a > \bar{V}_0 \\ &\ge \mathbb{E}V(x(t+\theta),t+\theta,r(t+\theta)) \ge \min_{1\le k\le N} \mathbb{E}V(\phi(\theta),t+\theta,k),\ \forall\,\theta \in [-\tau,0]. \end{aligned}$$

This, by condition (5), implies $\mathbb{E}\mathcal{L}V(x_t,t,r(t)) \le -\alpha_\lambda$ a.e. on $[t_0,t_1]$. Consequently, by (10), we have $\mathbb{E}V(x(t_1),t_1,r(t_1)) \le \bar{V}_0 - \alpha_\lambda\tilde{\tau} < 0$, which contradicts the property of $\mathbb{E}V(x(t),t,r(t)) \ge 0$ for all $t \ge 0$. So we must have $\tau_1 \le t_1$. Let $t_{1a} = \inf\{t > \tau_1 : \mathbb{E}V(x(t),t,r(t)) > M_1\}$. If $t_{1a} < \infty$, then there are constants t_{1b} and t_{1c} such that $t_1 \le t_{1b} \le t_{1a} < t_{1c}$ and

$$\begin{cases} \mathbb{E}V(x(t),t,r(t)) = M_1, & t = t_{1b}; \\ M_1 < \mathbb{E}V(x(t),t,r(t)) < M_1 + a, & t_{1b} < t \le t_{1c}. \end{cases} \tag{16}$$

Similarly, by (10) and (5), we find a contradiction and hence have (13) for $j = 1$.

Define $\tau_j = \inf\{t \ge t_{j-1} : \mathbb{E}V(x(t),t,r(t)) \le M_j\}$ for $j = 2,3,\cdots,J$. By the same type of reasoning, we have $\mathbb{E}V(x(t),t,r(t)) \le M_j$ for all $t \ge t_j$ and $j = 2,3,\cdots,J$. Particularly, $\mathbb{E}V(x(t),t,r(t)) \le M_J = \mu_w^{-1}(2\alpha_\lambda)$ for all $t \ge t_J$. By Jensen's inequality, we have

$$\mathbb{E}|x(t)|^p \le \gamma(||u_d||_\infty), \quad \forall\ t \ge t_J \tag{17}$$

where $\gamma(\cdot) = u^{-1}(\mu_w^{-1}(2\lambda(\cdot)))$.

Let $k = \frac{\bar{V}_0 - M_{J-1}}{t_{J-1}}$. Choose $\tilde{\beta} \in \mathcal{KL}$ such that $\tilde{\beta}(\bar{V}_0,t) \ge 2\bar{V}_0 - kt$ for all $0 \le t \le t_J$. So we have $\mathbb{E}V(x(t),t,r(t)) \le \tilde{\beta}(\bar{V}_0,t)$ for all $0 \le t \le t_J$, which implies

$$\mathbb{E}|x(t)|^p \le u^{-1}(\tilde{\beta}(\bar{V}_0,t)) = \beta(\mathbb{E}||\xi||^p,t), \quad \forall\ 0 \le t \le t_J \tag{18}$$

where $\beta(\cdot,\cdot) = u^{-1}(\tilde{\beta}(u(\cdot),\cdot))$ is a $\mathcal{KL}$ function. This completes the proof.

Remark 3.1: Obviously, inequality (3) implies that system (1) with $u_d(t) \equiv 0$ is globally pth moment asymptotically stable. Moreover, it is not difficult to show that if $|u(t)| \to 0$ as $t \to \infty$, so does $\mathbb{E}|x(t)|^p$ (see, e.g., Exercise 4.58, [11]). Therefore, by Theorem 3.1, it is easy to find that the HSDS, considered in Example 2.1 [41] but with mode-dependent and time-varying delay $\tilde{\tau} : R_+ \times S \to [0, \tau]$, is mean-quare asymptotically stable while the results in [41] do not work.

Remark 3.2: It is noted that inequality (6) removes the maximum operator on the right-hand side of corresponding conditions in the existing results (see Theorem 2.1, [23] and Theorem 3.2, [9]), which makes Theorem 3.1 less conservative but more applicable (see Example 4.1).

IV. Application and Example

Hybrid stochastic delay systems (HSDSs) described with stochastic differential delay equations with Markovian switching are an important class of HSRSs that are frequently used in engineering. As an illustrative example of applications of our new result, we consider the following HSDE

$$\mathrm{d}x(t) = F(x(t), x(t - \delta(t, r(t))), t, r(t), u_d(t))\mathrm{d}t + G(x(t), x(t - \delta(t, r(t))), t, r(t), u_d(t))\mathrm{d}B(t) \tag{19}$$

on $t \geq 0$, where $\delta : R_+ \times S \to [0, \tau]$ is Borel measurable while $F : R^n \times R^n \times R_+ \times S \times R^l \to R^n$ and $G : R^n \times R^n \times R_+ \times S \times R^l \to R^{n\times m}$ are measurable functions with $F(0, 0, t, i, 0) \equiv 0$ and $g(0, 0, t, i, 0) \equiv 0$ for all $t \geq 0$ and $i \in S$. Actually, this is a special case of equation (1) when $f(\phi, t, i, u_d) = F(\phi(0), \phi(-\delta(t, i)), t, i, u_d)$ and $g(\phi, t, i, u_d) = G(\phi(0), \phi(-\delta(t, i)), t, i, u_d)$ for $(\phi, t, i) \in C([-\tau, 0]; R^n) \times R_+ \times S \times R^l$while the operator $\mathcal{L}$ defined in (2) becomes from $R^n \times R^n \times R_+ \times S$ to R as

$$\begin{aligned}\mathcal{L}V(x, y, t, i) &= V_t(x, t, i) + V_x(x, t, i)F(x, y, t, i, u_d)\\ &\quad + \frac{1}{2} trace\left[G^T(x, y, t, i, u_d)V_{xx}(x, t, i)G(x, y, t, i, u_d)\right] + \sum_{j=1}^{N} \gamma_{ij} V(x, t, j).\end{aligned} \tag{20}$$

Let us use Theorem 3.1 to establish a useful criterion for system (19).

Theorem 4.1: Let $p > 0$, $u \in V\mathcal{K}_\infty$, $v \in \mathcal{K}_\infty$, $\lambda \in \mathcal{K}$ and $\kappa_{0i} \geq \kappa_{1i} \geq 0$, $i \in S$. Assume that there exists a function $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$ such that inequality (4) holds and, moreover,

$$\mathcal{L}V(x, y, t, i) \leq \lambda(|u_d(t)|) - \hat{\zeta}(x, i) - \kappa_{0i} V(x, t, i) + \kappa_{1i} \min_{1\leq k\leq N} V(y, t - \delta(t, i), k) \tag{21}$$

for all $(x, y, t, i) \in R^n \times R^n \times R_+ \times S$, where $\hat{\zeta} : R^n \times S \to R$ is a function such that there is $\hat{w} \in \mathcal{K}$ with $\hat{\zeta}(x, i) \geq \hat{w}(|x|)$ for all $i \in S$ and $\lim_{|x| \to \infty} \hat{w}(|x|)/v(|x|^p) > 0$. Then system (19) is pth moment ISS.

Proof For any $i \in S$, let

$$w(x, i) = \frac{1}{1 + \kappa_{0i}} \hat{\zeta}(x, i) \quad \text{and} \quad q(x, t, i) = V(x, t, i) + w(x, i) \tag{22}$$

in inequalities (5) and (6). By inequality (21) and Fatou's lemma, we have

$$\begin{aligned}
\mathbb{E}\mathcal{L}V(x, y, t, i) &\leq \lambda(|u_d(t)|) - \mathbb{E}\hat{\zeta}(x, i) - \kappa_{0i}\mathbb{E}V(x, t, i) + \kappa_{1i}\,\mathbb{E}\left[\min_{1 \leq k \leq N} V(y, t - \delta(t, i), k)\right] \\
&\leq \lambda(|u_d(t)|) - \kappa_{0i}(\mathbb{E}V(x, t, i) + \mathbb{E}w(x, i)) + \kappa_{1i} \min_{1 \leq k \leq N} \mathbb{E}V(y, t - \delta(t, i), k) - \mathbb{E}w(x, i) \\
&\leq \lambda(|u_d(t)|) - (\kappa_{0i} - \kappa_{1i})(\mathbb{E}V(x, t, i) + \mathbb{E}w(x, i)) - \mathbb{E}w(x, i) \\
&\leq \lambda(|u_d(t)|) - \mathbb{E}w(x, i)
\end{aligned}$$

for all $t \geq 0$, $i \in S$ and $x_t \in L^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$ satisfying condition (6) with function $q(x, t, i)$ defined in (22), i.e., $\min_{k \in S} \mathbb{E}V(y, t - \delta(t, i), k) < \mathbb{E}V(x, t, i) + \mathbb{E}w(x, i)$. Moreover, $\bar{w}(\cdot) = \zeta(\cdot) = \frac{1}{1+\kappa}\hat{w}(\cdot)$ satisfy the properties required in (5) and (6). By Theorem 3.1, inequality (3) holds for system (19).

To compare with the existing result in [9], let us consider the following example.

Example 4.1 Let $B(t)$ be a scalar Brownian motion. Let $r(t)$ be a right-continuous Markovian chain independent of $B(t)$ and taking values in $S = \{1, 2\}$ with generator $\Gamma = (\gamma_{ij})_{2 \times 2} = \left(\begin{smallmatrix} -1 & 1 \\ 2 & -2 \end{smallmatrix}\right)$. Consider a scalar uncertain stochastic delay system with Markovian switching of the form

$$\mathrm{d}x(t) = f(x(t), t, r(t))\mathrm{d}t + g(x(t - \delta(t, r(t))), t, r(t))\mathrm{d}B(t) \tag{23}$$

on $t \geq 0$, where $\delta : R_+ \times S \to [-\tau, 0]$ is a continuous but non-differentiable function with respect to t and

$$\begin{aligned}
f(x, t, 1) &= \frac{1}{4}x - \frac{1}{8}|x|\sqrt[3]{x}, \quad f(x, t, 2) = -bx - \frac{1}{10}x^3, \\
g(y, t, 1) &= \frac{1}{4}y\cos t, \quad g(y, t, 2) = \sqrt{2}y\sin t.
\end{aligned}$$

with $x = x(t)$, $y = x(t - \delta(t, r(t)))$ and positive constant b.

It is noted that the existing results [21], [23], [39], [41], [42] can not be applied to system (23), which has mode-dependent and time-varying delay $\delta(t, r(t))$. Observe that

$$2xf(x,t,1) \leq \frac{1}{2}x^2 - \frac{1}{4}|x|^{\frac{7}{3}}, \quad 2xf(x,t,2) \leq -2bx^2 - \frac{1}{5}x^4,$$
$$g^2(y,t,1) \leq \frac{1}{16}y^2, \quad g^2(y,t,2) \leq 2y^2.$$

To examine the stability of system (23), we construct a Lyapunov function candidate $V: R \times S \rightarrow R_+$ as $V(x,i) = \alpha_i x^2$ with $\alpha_2 = 1$ and $\alpha_1 > 0$ to be determined. By computation, we have

$$\mathcal{L}V(x,y,t,1) \leq -\frac{\alpha_1}{4}|x|^{\frac{7}{3}} - [\frac{\alpha_1}{2} - 1]x^2 + \frac{\alpha_1}{16}y^2, \tag{24}$$
$$\mathcal{L}V(x,y,t,2) \leq -\frac{1}{5}x^4 - (2 + 2b - 2\alpha_1)x^2 + 2y^2. \tag{25}$$

According to Theorem 4.2 in [9], inequalities (24) and (25) give

$$\lambda_{01} = \frac{1}{2} - \frac{1}{\alpha_1}, \quad \lambda_{11} = \frac{\alpha_1}{16}, \quad \lambda(s,1) = \frac{1}{4\sqrt[7]{\alpha_1}}s^{\frac{7}{6}};$$
$$\lambda_{02} = \frac{2(1+b)}{\alpha_1} - 2, \quad \lambda_{12} = 2, \quad \lambda(s,2) = \frac{1}{5\alpha_1^2}s^2.$$

Inequalities $\lambda_{01} \geq \lambda_{11}$ and $\lambda_{02} \geq \lambda_{12}$ yield $\alpha_1 = 4$ and $b \geq 7$. Then, by Theorem 4.2 in [9], system (23) is mean-square asymptotically stable if $b \geq 7$. However, for inequalities (24) and (25), we have

$$\kappa_{01} = \frac{1}{2} - \frac{1}{\alpha_1}, \quad \kappa_{11} = \frac{\alpha_1}{16}, \quad \hat{\zeta}(x,1) = \frac{\alpha_1}{4}|x|^{\frac{7}{3}};$$
$$\kappa_{02} = 2(1 + b - \alpha_1), \quad \kappa_{12} = 2, \quad \hat{\zeta}(x,2) = \frac{1}{5}x^4.$$

Inequalities $\kappa_{01} \geq \kappa_{11}$ and $\kappa_{02} \geq \kappa_{12}$ imply $\alpha_1 = 4$ and $b \geq 4$. By Theorem 4.1, the sufficient condition for mean-square asymptotical stability of system (23) is $b \geq 4$. Note that, when $4 \leq b < 7$, Theorem 4.2 in [9] does not work while Theorem 4.1 is still applicable to system (23). This shows Theorem 4.1 is more applicable.

V. Conclusion

This note improves an existing result in [9] and develops a Razumikhin-type theorem on input-to-state stability of HSRSs in pth $(p > 0)$ moment sense. It is seen that this improved result is less conservative but more applicable (see Remark 3.1, Remark 3.2 and Example 4.1).

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