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POPULATION DYNAMICAL BEHAVIOR OF NON-AUTONOMOUS LOTKA-VOLTERRA COMPETITIVE SYSTEM WITH RANDOM PERTURBATION

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Abstract. In this paper, we consider a non-autonomous stochastic Lotka-Volterra competitive system $dx_i(t) = x_i(t)[(b_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t))dt + \sigma_i(t)dB_i(t)]$, where $B_i(t)$ $(i = 1, 2, \cdots, n)$ are independent standard Brownian motions. Some dynamical properties are discussed and the sufficient conditions for the existence of global positive solutions, stochastic permanence, extinction as well as global attractivity are obtained. In addition, the limit of the average in time of the sample paths of solutions is estimated.

1. **Introduction.** It is an usual phenomena in nature that many species compete for limited resources or in some way inhibit others' growth. It is therefore very important to study the competitive models for multi-species. As we know, one of the famous models for population dynamics is the classical Lotka–Volterra competitive system which has received great attention and has been studied extensively owing to its theoretical and practical significance. The classical non-autonomous Lotka–Volterra competitive system can be expressed as follows

$$\dot{x}_i(t) = x_i(t)\left(b_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t)\right),\ i = 1, \cdots, n, \tag{1}$$

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where $x_i(t)$ denotes the population size of the ith species at time t and all parameter functions $b_i(t)$ and $a_{ij}(t)$ are *nonnegative*. In mathematical ecology, equation (1) describes an n-species dynamical system in which each individual competes with others for the limited resources.

In [2], introducing a notation of the upper and lower averages of a function, Ahmad and Lazer obtained sufficient conditions which guarantee the permanence and global attractivity of system (1). Zhao et al. [34] obtained some excellent results which generalized the main results of [2]. Several other authors also investigated the uniform persistence and extinction for partial species, see, for example, Chen [7], Ahmad [1], Montesf de Oca and Zeeman [28], Teng [32] and Zhao [33]. In particular, the books by Golpalsamy [17] and Kuang [21] are good references in this area.

On the other hand, population systems are often subject to environmental noise. It is therefore useful to reveal how the noise affects the population systems. As a matter of fact, stochastic population systems have recently been studied by many authors, for example,[4]-[6], [12]-[14], [20, 23, 25, 26, 30]. In particular, Mao, Marion and Renshaw [25, 26] revealed that the environmental noise can suppress a potential population explosion while Mao [23] showed that different structures of environmental noise may have different effects on the population systems. However, almost all known stochastic models assume that the growth rate and the carrying capacity of the population are independent of time t. In contrast, the natural growth rates of many populations vary with t in real situation, for example, due to the seasonality. So far, a few results for non-autonomous population systems with random perturbation exist in the literature and, to the best of our knowledge, there are two papers [10, 11] which deal with *1-species* non-autonomous logistic equation with random perturbation. This is of course due to the fact that non-autonomous stochastic population systems for multiple-species are much harder than autonomous ones.

In this paper we consider the situation of the parameter perturbation. Recall that the parameter function $b_i(t)$ of equation (1) represents the intrinsic growth rate of species i at time t. In practice we usually estimate it by an average value plus an error term. If we still use $b_i(t)$ to denote the average growth rate at time t, then the intrinsic growth rate depending on time t becomes

$$b_i(t) \to b_i(t) + \sigma_i(t)\dot{B}_i(t),$$

where $\dot{B}_i(t)$ is a white noise and $\sigma_i^2(t)$ represents the intensity of the noise. Then the stochastically perturbed system can be described by the Itô equation

$$dx_i(t) = x_i(t)\left[\left(b_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t)\right)dt + \sigma_i(t)dB_i(t)\right], \quad i = 1, 2, \cdots, n, \tag{2}$$

where $B_i(t)$, $1 \le i \le n$, are independent standard Brownian motions, and $b_i(t)$, $a_{ij}(t)$, $\sigma_i(t)$ are all continuous bounded nonnegative functions on $[0, +\infty)$.

Since equation (2) describes a stochastic population dynamics, it is critical for the solution to remain positive and not to explode to infinity in a finite time. In order for a stochastic differential equation to have an unique global (i.e. no explosion in a finite time) solution for any initial value, the coefficients of the equation are usually required to satisfy the linear growth condition and local Lipschitz condition (see e.g. [3, 8, 18]). However, the coefficients of equation(2) do not satisfy the linear growth condition, though they are locally Lipschitz continuous. We may therefore wonder if the solution of equation (2) may explode at a finite time. We shall show this is not possible in section 2.

In the study of stochastic population systems, the stochastic permanence, which means that the population system will survive forever, is one of the most important and interesting topics. In section 3, we discuss this situation and prove that when the noise is small enough the population system is stochastically permanent. Our result is not only more general than [11] but the conditions imposed are weaker than those used in [11] for the 1-dimension case. In section 4, we will discuss the problem of extinction. We will show that a sufficiently large noise will force every species become extinct. In section 5, we will prove that the limit of the average in time of the sample paths of the solutions is bounded with probability one and give an estimation for it. Finally, in section 6, we will study the global attractivity.

The key method used in this paper is the analysis of Lyapunov functions. This Lyapunov functions analysis for stochastic differential equations was developed by Khasminskii (see e.g. [18]) and has been used by many authors (see e.g. [3, 8, 15, 24, 27, 29, 31]).

2. **Positive and Global Solutions.** Throughout this paper, unless otherwise specified, let $(\Omega,\ F,\ \{F_t\}_{t\geq 0},\ P)$ be a complete probability space with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and increasing while F_0 contains all P-null sets). Let $B_i(t)$ $(i = 1, \cdots, n)$ denote the independent standard Brownian motions defined on this probability space. We also denote by R^n_+ the positive cone in R^n, that is $R^n_+ = \{x \in R^n : x_i > 0 \text{ for all } 1 \leq i \leq n\}$, and denote by $\bar{R}^n_+$ the nonnegative cone in R^n, that is $\bar{R}^n_+ = \{x \in R^n : x_i \geq 0 \text{ for all } 1 \leq i \leq n\}$. If $x \in R^n$, its norm is denoted by $|x| = (\sum\limits_{i=1}^{n} x_i^2)^{\frac{1}{2}}$.

If $f(t)$ is a continuous bounded function on $[0, +\infty)$, then define

$$f^u = \sup_{t\in[0,+\infty)} f(t),\ f^l = \inf_{t\in[0,+\infty)} f(t),$$

$$[f(t)]^+ = \begin{cases} f(t), & f(t) > 0; \\ 0, & f(t) \leq 0, \end{cases} \quad [f(t)]^- = \begin{cases} -f(t), & f(t) < 0; \\ 0, & f(t) \geq 0. \end{cases}$$

For any constant sequence $\{c_{ij}\},\ (1 \leq i \leq n, 1 \leq j \leq n)$ define

$$(\breve{c_{ij}}) = \max_{1\leq i,j\leq n} c_{ij},\ (\breve{c_{ij}})_i = \max_{1\leq i\leq n} c_{ij},\ (\breve{c_{ij}})_j = \max_{1\leq j\leq n} c_{ij},\ (\breve{c_{ii}}) = \max_{1\leq i\leq n} c_{ii},$$

$$(\hat{c_{ij}}) = \min_{1\leq i,j\leq n} c_{ij},\ (\hat{c_{ij}})_i = \min_{1\leq i\leq n} c_{ij},\ (\hat{c_{ij}})_j = \min_{1\leq j\leq n} c_{ij},\ (\hat{c_{ii}}) = \min_{1\leq i\leq n} c_{ii}.$$

The following theorem is fundamental in this paper.

Theorem 2.1. *For any given initial value $x(0) \in R^n_+$, there is an unique solution $x(t)$ to equation (2) on $t \geq 0$ and the solution will remain in R^n_+ with probability 1, namely $x(t) \in R^n_+$ for all $t \geq 0$ almost surely.*

The proof is a modification of that for the autonomous case (see e.g. [5, 6, 23] but for the completeness of the paper we will give it in Appendix A.

Specially, let us consider the 1-dimensional non-autonomous stochastic logistic equation

$$dN(t) = N(t)\left[(b(t) - a(t)N(t))\,dt + \sigma(t)dB(t)\right],\ \ t \geq 0. \tag{3}$$

Here $B(t)$ is the 1-dimensional standard Brownian motion, $N(0) \in R_+$. Moreover, $b(t),\ a(t)$ and $\sigma(t)$ are continuous bounded nonnegative functions defined on $[0, \infty)$. By Theorem 2.1, we observe that for any given initial value $N(0) \in R_+$, there is an unique solution $N(t)$ to equation (3) on $t \geq 0$ and the solution will remain in R_+

with probability 1. This is a better result than the corresponding one in [10] which requires that $a(t) > 0$ and $b(t) > 0$.

3. **Stochastic Permanence.** Theorem 2.1 shows that the solution of equation (2) will remain in the positive cone R_+^n. This nice property provides us with a great opportunity to discuss how the solution varies in R_+^n in more detail. Let us impose a simple hypothesis.

Assumption 1. *$a_{ii}^l > 0$ $(1 \le i \le n)$.*

The following lemma shows that under this simple assumption, the solutions of equation(2) are asymptotically bounded in any pth moment.

Lemma 3.1. *Let Assumption 1 hold and $p > 0$. Then for any initial value $x(0) \in R_+^n$, the solution $x(t)$ of equation(2) obeys*

$$\limsup_{t\to\infty} E(|x(t)|^p) \le K(p), \tag{4}$$

where $K(p)$ is independent of $x(0)$ and defined by

$$K(p) := n^{\frac{p}{2}} \sum_{i=1}^n K_i(p) \text{ and } K_i(p) := \begin{cases} \left(\dfrac{b_i^u}{a_{ii}^l}\right)^p, & \text{for} \quad 0 < p < 1; \\ \left[\dfrac{b_i^u + \frac{p-1}{2}(\sigma_i^u)^2}{a_{ii}^l}\right]^p, & \text{for} \quad p \ge 1. \end{cases} \tag{5}$$

The proof of this lemma is rather standard and hence is omitted. By this lemma, we can easily obtain the stochastic ultimate boundedness which is one of the important topics in population systems and is defined as follows.

Definition 3.2. The solutions of equation (2) are said to be *stochastically ultimately bounded*, if for any $\epsilon \in (0,\ 1)$, there is a positive constant $\chi(= \chi(\epsilon))$, such that for any initial value $x(0) \in R_+^n$, the solution of equation (2) has the property that

$$\limsup_{t\to\infty} P\{|x(t)| > \chi\} < \epsilon.$$

By Chebyshev's inequality and Lemma 3.1, the following result is straightforward.

Theorem 3.3. *The solutions of equation (2) are stochastically ultimately bounded under Assumption 1.*

The following lemma gives a pathwise estimation for the solutions which will be used later.

Lemma 3.4. *Under Assumption 1, for any given initial value $x(0) \in R_+^n$, the solution $x(t)$ of equation (2) has the properties that*

$$\limsup_{t\to\infty} E\left(\sup_{t\le r\le t+1} |x(r)|\right) \le [1 + (\breve{b_i^u})]n^{-\frac{1}{2}} K(1) + 3(\breve{\sigma_i^u})[K(2)]^{\frac{1}{2}} \tag{6}$$

and

$$\limsup_{t\to\infty} \frac{\log(|x(t)|)}{\log t} \le 1 \ \ a.s. \tag{7}$$

The proof is somehow standard so we only give a brief one in Appendix B. Let us now impose another hypothesis.

Assumption 2. *Define*

$$r_i(t) = b_i(t) - \frac{1}{2}\sigma_i^2(t), \quad t \geq 0,\ 1 \leq i \leq n \tag{8}$$

and assume that $(\hat{r_i^l}) > 0$.

Lemma 3.5. *Under Assumption 2, for any given initial value* $x(0) \in R_+^n$*, the solution* $x(t)$ *of equation (2) has the properties that*

$$\limsup_{t\to\infty} E(\frac{1}{|x(t)|^\theta}) \leq H \tag{9}$$

and

$$\liminf_{t\to\infty} \frac{\log(|x(t)|)}{\log t} \geq -\frac{(\breve{\sigma_i^u})^2}{2(\hat{r_i^l})} \ \ a.s \tag{10}$$

where θ *is an arbitrary positive constant satisfying*

$$\theta(\breve{\sigma_i^u})^2 < 2(\hat{r_i^l}), \tag{11}$$

and

$$H := \frac{\theta n^\theta (c_2 + 4c_1c_3)}{8n\kappa c_1} \max\left\{1,\ \left(\frac{2c_1 + c_2 + \sqrt{c_2^2 + 4c_1c_2}}{2c_1}\right)^{\theta-2}\right\}, \tag{12}$$

in which κ *is an arbitrary positive constant satisfying*

$$0 < \frac{2n\kappa}{\theta} < 2(\hat{r_i^l}) - \theta(\breve{\sigma_i^u})^2, \tag{13}$$

while

$$c_1 := 2(\hat{r_i^l}) - \theta(\breve{\sigma_i^u})^2 - \frac{2n\kappa}{\theta} > 0,\ \ c_2 := 2n[(\breve{a_{ij}^u}) + (\breve{\sigma_i^u})^2 + \frac{2\kappa}{\theta}] > 0,$$

$$c_3 := 2n[(\breve{a_{ij}^u}) + \frac{\kappa}{\theta}] > 0.$$

Proof. Define $V(x) = \sum_{i=1}^n x_i$ for $x \in R_+^n$ and

$$U(t) = \frac{1}{V(x(t))} \text{ on } t \geq 0.$$

By the Itô formula, we have

$$\begin{aligned} &dU(t) \\ &= \left[-U^2(t)\sum_{i=1}^n x_i(t)\left(b_i(t) - \sum_{j=1}^n a_{ij}(t)x_j(t)\right) + U^3(t)\sum_{i=1}^n(\sigma_i(t)x_i(t))^2\right]dt \\ &\quad -U^2(t)\sum_{i=1}^n \sigma_i(t)x_i(t)dB_i(t). \end{aligned}$$

Under Assumption 2, we can certainly choose a positive constant θ such that it satisfies (11). Applying the Itô formula again, we have

$$\begin{aligned} d\left[(1+U(t))^\theta\right] &= \theta(1+U(t))^{\theta-2}J(t)dt \\ &- \theta(1+U(t))^{\theta-1}U^2(t)\sum_{i=1}^n \sigma_i(t)x_i(t)dB_i(t), \end{aligned} \tag{14}$$

where

$$J(t) = -(1+U(t))U^2(t)\sum_{i=1}^{n} x_i(t)\left(b_i(t)-\sum_{j=1}^{n} a_{ij}(t)x_j(t)\right) +(1+U(t))U^3(t)\sum_{i=1}^{n}(\sigma_i(t)x_i(t))^2+\frac{\theta-1}{2}U^4(t)\sum_{i=1}^{n}(\sigma_i(t)x_i(t))^2.$$

It is not difficult to estimate

$$J(t)\le -\frac{1}{2n}[2(\hat{r_i^l})-\theta(\breve{\sigma_i^u})^2]U^2(t)+[(\breve{a_{ij}^u})+(\breve{\sigma_i^u})^2]U(t)+(\breve{a_{ij}^u}).$$

Substituting this into (14) yields

$$\begin{aligned} & d\left[(1+U(t))^\theta\right] \\ \le\ & \theta(1+U(t))^{\theta-2} \\ & \times\left\{-\frac{1}{2n}[2(\hat{r_i^l})-\theta(\breve{\sigma_i^u})^2]U^2(t)+[(\breve{a_{ij}^u})+(\breve{\sigma_i^u})^2]U(t)+(\breve{a_{ij}^u})\right\}dt \\ & -\theta(1+U(t))^{\theta-1}U^2(t)\sum_{i=1}^{n}\sigma_i(t)x_i(t)dB_i(t). \end{aligned} \tag{15}$$

Now, choose $\kappa>0$ sufficiently small for (13) to hold. Then, by the Itô formula,

$$\begin{aligned} & d\left[e^{\kappa t}(1+U(t))^\theta\right] \\ & =\kappa e^{\kappa t}(1+U(t))^\theta dt+e^{\kappa t}d\left[(1+U(t))^\theta\right] \\ & \le e^{\kappa t}(1+U(t))^{\theta-2}\left\{\kappa(1+U(t))^2-\frac{\theta}{2n}[2(\hat{r_i^l})-\theta(\breve{\sigma_i^u})^2]U^2(t)\right. \\ & \quad\left.+\theta[(\breve{a_{ij}^u})+(\breve{\sigma_i^u})^2]U(t)+\theta(\breve{a_{ij}^u})\right\}dt \\ & \quad-\theta e^{\kappa t}(1+U(t))^{\theta-1}U^2(t)\sum_{i=1}^{n}\sigma_i(t)x_i(t)dB_i(t) \\ & =\frac{\theta}{2n}e^{\kappa t}(1+U(t))^{\theta-2}\left\{-U^2(t)[2(\hat{r_i^l})-\theta(\breve{\sigma_i^u})^2-\frac{2n\kappa}{\theta}]\right. \\ & \quad\left.+2n[(\breve{a_{ij}^u})+(\breve{\sigma_i^u})^2+\tfrac{2\kappa}{\theta}]U(t)+2n(\breve{a_{ij}^u})+\frac{2n\kappa}{\theta}\right\}dt \\ & \quad-\theta e^{\kappa t}(1+U(t))^{\theta-1}U^2(t)\sum_{i=1}^{n}\sigma_i(t)x_i(t)dB_i(t). \end{aligned}$$

It is easy to see that

$$\begin{aligned} & \frac{\theta}{2n}(1+U(t))^{\theta-2}\left\{-U^2(t)[2(\hat{r_i^l})-\theta(\breve{\sigma_i^u})^2-\frac{2n\kappa}{\theta}]\right. \\ & \quad\left.+2n[(\breve{a_{ij}^u})+(\breve{\sigma_i^u})^2+\frac{2\kappa}{\theta}]U(t)+2n(\breve{a_{ij}^u})+\frac{2n\kappa}{\theta}\right\}\le H_1, \end{aligned} \tag{16}$$

on $U(t)>0$, where

$$H_1:=\frac{\theta(c_2+4c_1c_3)}{8nc_1}\max\left\{1,\ \left(\frac{2c_1+c_2+\sqrt{c_2^2+4c_1c_2}}{2c_1}\right)^{\theta-2}\right\},$$

and c_1-c_3 have been defined in the statement of the theorem. Thus

$$d\left[e^{\kappa t}(1+U(t))^\theta\right]\le H_1e^{\kappa t}dt-\theta e^{\kappa t}(1+U(t))^{\theta-1}U^2(t)\sum_{i=1}^{n}\sigma_i(t)x_i(t)dB_i(t).$$

This implies

$$E\left[e^{\kappa t}(1+U(t))^{\theta}\right] \leq (1+U(0))^{\theta} + \frac{H_1}{\kappa}e^{\kappa t}.$$

Then

$$\limsup_{t\to\infty} E\left[U^{\theta}(t)\right] \leq \limsup_{t\to\infty} E\left[(1+U(t))^{\theta}\right] \leq \frac{H_1}{\kappa}. \tag{17}$$

For $x(t) \in R^n_+$, note that

$$\left(\sum_{i=1}^{n} x_i(t)\right)^{\theta} \leq \left(n \max_{1\leq i\leq n} x_i(t)\right)^{\theta} = n^{\theta}\left(\max_{1\leq i\leq n} x_i^2(t)\right)^{\frac{\theta}{2}} \leq n^{\theta}|x(t)|^{\theta}. \tag{18}$$

Consequently,

$$\limsup_{t\to\infty} E(\frac{1}{|x(t)|^{\theta}}) \leq n^{\theta}\frac{H_1}{\kappa} = H,$$

which is the required assertion (9).

Moreover, using (16), we observe from (15) that

$$d\left[(1+U(t))^{\theta}\right] \leq H_1 dt - \theta(1+U(t))^{\theta-1}U^2(t)\sum_{i=1}^{n}\sigma_i(t)x_i(t)dB_i(t).$$

This implies that

$$\begin{aligned} E\left[\sup_{t\leq r\leq t+1}(1+U(r))^{\theta}\right] &\leq E\left[(1+U(t))^{\theta}\right] + H_1 \\ &\quad + E\left(\sup_{t\leq r\leq t+1}|\int_t^r \theta(1+U(s))^{\theta-1}U^2(s)\sum_{i=1}^{n}\sigma_i(s)x_i(s)dB_i(s)|\right). \end{aligned} \tag{19}$$

But, by the well-known Burkholder-Davis-Gundy inequality and the Hölder inequality, we can show that

$$\begin{aligned} & E\left(\sup_{t\leq r\leq t+1}|\int_t^r \theta(1+U(s))^{\theta-1}U^2(s)\sum_{i=1}^{n}\sigma_i(s)x_i(s)dB_i(s)|\right) \\ \leq\ & \frac{1}{2}E\left(\sup_{t\leq r\leq t+1}(1+U(r))^{\theta}\right) + 9\theta^2(\breve{\sigma_i^u})^2 E\left(\int_t^{t+1}(1+U(s))^{\theta}ds\right). \end{aligned}$$

Substituting this into (19) gives

$$\begin{aligned} & E\left[\sup_{t\leq r\leq t+1}(1+U(r))^{\theta}\right] \\ & \leq 2E\left[(1+U(t))^{\theta}\right] + 2H_1 + 18\theta^2(\breve{\sigma_i^u})^2 E\left(\int_t^{t+1}(1+U(s))^{\theta}ds\right). \end{aligned}$$

Letting $t\to\infty$ and using (17) we obtain that

$$\limsup_{t\to\infty} E\left[\sup_{t\leq r\leq t+1}(1+U(r))^{\theta}\right] \leq 2[1+9\theta^2(\breve{\sigma_i^u})^2]\frac{H_1}{\kappa} + 2H_1. \tag{20}$$

Using (18) and recalling the definition $U(t)$,

$$\limsup_{t\to\infty} E\left[\sup_{t\leq r\leq t+1}\frac{1}{|x(r)|^{\theta}}\right] \leq 2[1+9\theta^2(\breve{\sigma_i^u})^2]n^{\theta}\frac{H_1}{\kappa} + 2n^{\theta}H_1. \tag{21}$$

From this we can show, in the same way as (7) was proved, that

$$\limsup_{t\to\infty} \frac{\log\left(\dfrac{1}{|x(t)|^\theta}\right)}{\log t} \le 1 \text{ a.s.}$$

which further implies

$$\liminf_{t\to\infty} \frac{\log(|x(t)|}{\log t} \ge -\frac{1}{\theta} \text{ a.s.}$$

But this holds for any θ that obeys (11). We must therefore have the assertion (10). □

We are now in the position to show the stochastic permanence which is defined below.

Definition 3.6. Equation (2) is said to be stochastically permanent if for any $\varepsilon \in (0,\ 1)$, there exists a pair of positive constants $\delta = \delta(\epsilon)$ and $\chi = \chi(\epsilon)$ such that for any initial value $x(0) \in R^n_+$, the solution obeys

$$\liminf_{t\to\infty} P\left\{|x(t)| \le \chi\right\} \ge 1-\epsilon \quad \text{and} \quad \liminf_{t\to\infty} P\left\{|x(t)| \ge \delta\right\} \ge 1-\epsilon.$$

Theorem 3.7. *Under Assumptions 1 and 2, equation (2) is stochastically permanent.*

The proof is a simple application of the Chebyshev inequality and Lemmas 3.1 and 3.5. Applying our theorem above to equation (3), we immediately obtain the following result.

Corollary 1. *Suppose that $a^l > 0$ and $(b - \frac{1}{2}\sigma^2)^l > 0$. Then equation (3) is stochastically permanent.*

In [11], the authors show that equation (3) is stochastically permanent if $a(t)$, $b(t)$, and $\sigma(t)$ are continuous T-periodic functions, $a(t) > 0$, $b(t) > 0$ and $\min_{t\in[0,T]} a(t) > \max_{t\in[0,T]} \sigma^2(t)$. Obviously the conditions of Corollary 1 are much weaker than these.

It is also useful to point out that how the stochastic permanence relates to the classical concepts of recurrence and transience. It is easy to see that transience implies non-permanence while permanence implies non-transience. However, a system may be stochastically permanent but not recurrent. For instance, if the solutions of a system will converge to a bounded set in R^n_+ with probability one, it is then stochastic permanent but it is not recurrent clearly. Moreover, recurrence does not implies permanence either.

4. **Extinction.** In the previous sections we have showed that under certain conditions, the original non-autonomous equations (1) and the associated stochastic equation (2) behave similarly in the sense that both have positive solutions which will not explode to infinity in a finite time and, in fact, will be ultimately bounded and permanent. In other words, we show that under certain condition the noise will not spoil these nice properties. However, we will show in this section that if the noise is sufficiently large, the solution to the associated stochastic equation (2) will become extinct with probability one, although the solution to the original equation (1) may be persistent. For example, recall a simple case, namely the scalar logistic equation

$$dN(t) = N(t)(b - aN(t))dt, \ \ t \ge 0. \tag{22}$$

It is well known that if $b > 0,\ a > 0$, then its solution $N(t)$ is persistent for

$$\lim_{t\to\infty} N(t) = \frac{b}{a}.$$

However, consider its associated stochastic equation

$$dN(t) = N(t)[(b - aN(t))dt + \sigma dB(t)],\ \ t \geq 0, \tag{23}$$

where $\sigma > 0$. We will see from the following theorem that if $\sigma^2 > 2b$, then the solution to this stochastic equation will become extinct with probability one, namely

$$\lim_{t\to\infty} N(t) = 0 \text{ a.s.}$$

In other words, the following theorem reveals the important fact that the environmental noise may make the population extinct.

Theorem 4.1. *For any given initial value $x(0) \in R^n_+$, the solution $x(t)$ of equation (2) has the property that, for every $1 \leq i \leq n$,*

$$\limsup_{t\to\infty} \frac{\log x_i(t)}{t} \leq \limsup_{t\to\infty} \frac{1}{t}\int_0^t [b_i(s) - \frac{\sigma_i^2(s)}{2}]ds \ \ a.s. \tag{24}$$

Proof. By the Itô formula, for each $1 \leq i \leq n$, we derive from (2) that

$$d\log x_i(t) = \left[b_i(t) - \frac{\sigma_i^2(t)}{2} - \sum_{j=1}^n a_{ij}(t)x_j(t)\right] dt + \sigma_i(t)dB_i(t). \tag{25}$$

Hence,

$$\begin{aligned} & \log x_i(t) & (26)\\ = \ & \log x_i(0) + \int_0^t [b_i(s) - \frac{\sigma_i^2(s)}{2}]ds - \sum_{j=1}^n \int_0^t a_{ij}(s)x_j(s)ds + \int_0^t \sigma_i(s)dB_i(s) & \\ \leq \ & \log x_i(0) + \int_0^t [b_i(s) - \frac{\sigma_i^2(s)}{2}]ds + M(t), & (27) \end{aligned}$$

where $M(t)$ is a martingale defined by

$$M(t) = \int_0^t \sigma_i(t)dB_i(t).$$

The quadratic variation of this martingale is

$$\langle M, M\rangle_t = \int_0^t \sigma_i^2(s)ds \leq (\sigma_i^u)^2 t.$$

By the strong law of large numbers for martingales (see [24], [27]), we therefore have

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0 \text{ a.s.}$$

It finally follows from (27) by dividing t on the both sides and then letting $t \to \infty$ that

$$\limsup_{t\to\infty} \frac{\log x_i(t)}{t} \leq \limsup_{t\to\infty} \frac{1}{t}\int_0^t [b_i(s) - \frac{\sigma_i^2(s)}{2}]ds \text{ a.s.}$$

which is the required assertion (24).

$\square$

It follows from Theorem 4.1 immediately that

$$\limsup_{t\to\infty} \frac{\log x_i(t)}{t} \le r_i^u \text{ a.s.}$$

for all $1 \le i \le n$. This theorem also leads us to impose:

Assumption 3. $\limsup_{t\to\infty} \frac{1}{t}\int_0^t [b_i(s) - \frac{\sigma_i^2(s)}{2}]ds < 0$ *for all* $i = 1, \dots, n$.

Corollary 2. *Under Assumption 3, for any given initial value $x(0) \in R_+^n$, the solution $x(t)$ of equation (2) will become extinct (i.e. tend to zero) exponentially with probability one.*

In this corollary, the sample Lyapunov exponent of the solution is negative, but this may not be necessary in order for extinction to happen. To reveal this situation, let us recall a LaSalle-type theorem for stochastic differential equations. We should also point out that the result [29, Theorem 8.1] on convergence of solutions to a set may also be applied to this situation.

Consider the n-dimensional stochastic differential equation

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \tag{28}$$

on $t \ge 0$ with initial value $x(0) \in R^n$. Here $B(t) = (B_1(t), B_2(t), \dots, B_n(t))^T$ is an m-dimensional Brownian motion, $f : R^n \times \bar{R}_+ \to R^n$ and $g : R^n \times \bar{R}_+ \to R^{n\times m}$. The following hypothesis is imposed in [22]:

(*) For any initial value $x(0) \in R^n$, equation (28) has a unique solution denoted by $x(t, x(0))$ on $t \ge 0$. Moreover, for every $h > 0$, there is a $K_h > 0$ such that

$$|f(x,t)| \vee |g(x,t)| \le K_h$$

for all $t \ge 0$ and $x \in R^n$ with $|x| \le h$.

For each $V \in C^{2,1}(R^n \times \bar{R}_+; \bar{R}_+)$, define an operator LV from $R^n \times \bar{R}_+$ to R by

$$LV(x,t) = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}trace[g^T(x,t)V_{xx}(x,t)g(x,t)], \tag{29}$$

where

$$V_t(x,t) = \frac{\partial V(x,t)}{\partial t},\ V_x(x,t) = \left(\frac{\partial V(x,t)}{\partial x_1}, \dots, \frac{\partial V(x,t)}{\partial x_n}\right),$$

$$V_{xx}(x,t) = \left(\frac{\partial^2 V(x,t)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

Lemma 4.2. *([22]) Let condition (*) hold. Assume that there are functions $V \in C^{2,1}(R^n \times \bar{R}_+; \bar{R}_+)$, $\gamma \in L^1(\bar{R}_+; \bar{R}_+)$, and $w \in C(R^n; \bar{R}_+)$ such that*

$$LV(x,t) \le \gamma(t) - w(x),\ (x,t) \in R^n \times \bar{R}_+,$$

and

$$\lim_{|x|\to\infty} \inf_{0\le t<\infty} V(x,t) = \infty.$$

Then $Ker(w) \ne \emptyset$ and $\lim_{t\to\infty} d(x(t,x(0)), Ker(w)) = 0$ a.s. for every $x(0) \in R^n$.

To apply this lemma to study the problem of extinction, let us impose one more hypothesis:

Assumption 4. *There exist some constants $0 < \alpha_i \ll 1$, $i = 1, 2, \ldots, n$, such that*

$$\sum_{i=1}^{n} \int_0^{\infty} [b_i(t) - \frac{(1-\alpha_i)\sigma_i^2(t)}{2}]^+ dt < \infty.$$

Theorem 4.3. *Under Assumptions 1 and 4, for any given initial value $x(0) \in R_+^n$, the solution $x(t)$ of equation (2) has the property that*

$$\lim_{t\to\infty} |x(t)| = 0 \ a.s. \tag{30}$$

Proof. It is obvious that equation (2) satisfies the condition (*) of Lemma 4.2. By the Itô's formula, for the given constant α_i specified in Assumption 4, we compute

$$\begin{aligned}
& dx_i^{\alpha_i}(t) \\
= \;& \alpha_i x_i^{\alpha_i - 1}(t) dx_i(t) + \frac{\alpha_i(\alpha_i - 1)}{2} x_i^{\alpha_i - 2}(t)(dx_i(t))^2 \\
= \;& \alpha_i x_i^{\alpha_i} \left[\left(b_i(t) - \sum_{j=1}^{n} a_{ij}(t) x_j \right) dt + \frac{\alpha_i - 1}{2}\sigma_i^2(t) \right] dt + \alpha_i x_i^{\alpha_i} \sigma_i(t) dB_i(t) \\
= \;& \alpha_i x_i^{\alpha_i} \left(b_i(t) - \frac{1-\alpha_i}{2}\sigma_i^2(t) \right) dt - \alpha_i x_i^{\alpha_i} \sum_{j=1}^{n} a_{ij}(t) x_j dt + \alpha_i x_i^{\alpha_i} \sigma_i(t) dB_i(t) \\
\le \;& \alpha_i x_i^{\alpha_i} \left(b_i(t) - \frac{1-\alpha_i}{2}\sigma_i^2(t) \right) dt - \alpha_i a_{ii}(t) x_i^{\alpha_i + 1} dt + \alpha_i x_i^{\alpha_i} \sigma_i(t) dB_i(t),
\end{aligned} \tag{31}$$

where we have dropped t from $x_i(t)$. Define

$$Z_i(t) = e^{-\alpha_i \int_0^t \left(b_i(s) - \frac{1-\alpha_i}{2}\sigma_i^2(s) \right) ds},$$

and

$$\tilde{V}(x,t) = \sum_{i=1}^{n} Z_i(t) x_i^{\alpha_i}.$$

Obviously, under Assumption 4, $\tilde{V} \in C^{2,1}(R_+^n \times \bar{R}_+; \bar{R}_+)$ and

$$\lim_{|x|\to\infty} \inf_{0\le t<\infty} \tilde{V}(x,t) = \infty,$$

in addition, there exists an positive constant U_i such that $Z_i(t) \ge U_i$. By the Itô formula, we derive from (31)

$$\begin{aligned}
d\tilde{V}(x,t) \;&=\; -\sum_{i=1}^{n} \alpha_i Z_i(t) x_i^{\alpha_i}(t) \left(b_i(t) - \frac{1-\alpha_i}{2}\sigma_i^2(t) \right) dt + \sum_{i=1}^{n} Z_i(t) dx_i^{\alpha_i}(t) \\
&\le\; -\sum_{i=1}^{n} \alpha_i a_{ii}(t) Z_i(t) x_i^{\alpha_i+1}(t) dt + \sum_{i=1}^{n} \alpha_i Z_i(t) x_i^{\alpha_i} \sigma_i(t) dB_i(t) \\
&\le\; -\sum_{i=1}^{n} \alpha_i a_{ii}^l U_i x_i^{\alpha_i+1}(t) dt + \sum_{i=1}^{n} \alpha_i Z_i(t) x_i^{\alpha_i}(t) \sigma_i(t) dB_i(t).
\end{aligned} \tag{32}$$

Therefore

$$L\tilde{V}(x,t) \le -\sum_{i=1}^{n} \alpha_i a_{ii}^l U_i x_i^{\alpha_i+1}(t) =: -w(x(t)).$$

Obviously,

$$Ker(w) = \{(0, 0, \ldots, 0)^T\}.$$

By Lemma 4.2, we obtain the assertion (30). □

Corollary 3. *Under Assumptions 1, if $b_i \in L^1(\bar{R}_+; \bar{R}_+)$ for every $1 \le i \le n$, then the solutions of equation (2) will become extinct with probability one.*

5. **Asymptotic Boundedness of Integral Average.** Let us now begin to discuss the average in time of the underlying population.

Theorem 5.1. *Let Assumption 2 hold. If $(\hat{a^l_{ij}}) > 0$, for any initial value $x(0) \in R^n_+$, the solution $x(t)$ of equation (2) obeys*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t |x(s)|ds \le \frac{(\breve{r^u_i}) + (\breve{b^u_i})}{(\hat{a^l_{ij}})} \ \ a.s. \tag{33}$$

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t |x(s)|ds \ge \frac{(\hat{r^l_i})}{n^{\frac{3}{2}}(\breve{a^u_{ij}})} \ \ a.s. \tag{34}$$

where $r_i(t)$ is defined by (8).

Proof. Define $V(x) = \sum_{i=1}^n x_i$ for $x \in R^n_+$. It is easy to observe from inequality (7) of Lemma 3.4 and (10) of Lemma 3.5 that

$$\lim_{t\to+\infty} \frac{\log V(x(t))}{t} = 0 \text{ a.s.} \tag{35}$$

It is not difficult to show that

$$\begin{aligned} & d[\log V(x(t))] \qquad (36)\\ \le\ & \left[\frac{\sum\limits_{i=1}^n [b_i(t) - \frac{\sigma_i^2(t)}{2}]x_i^2 + \sum\limits_{i=1}^n \sum\limits_{j=1, j\ne i}^n b_j(t)x_ix_j}{\left(\sum\limits_{i=1}^n x_i\right)^2} - (\hat{a^l_{ij}})V(x(t))\right] dt \\ +\ & \frac{\sum\limits_{i=1}^n \sigma_i(t)x_i dB_i(t)}{\sum\limits_{i=1}^n x_i}. \qquad (37)\end{aligned}$$

By Assumption 2, we then see

$$d[\log V(x(t))] \le [(\breve{r^u_i}) + (\breve{b^u_i}) - (\hat{a^l_{ij}})V(x(t))]dt + \frac{\sum\limits_{i=1}^n \sigma_i(t)x_i(t)dB_i(t)}{\sum\limits_{i=1}^n x_i(t)}. \tag{38}$$

Hence

$$\begin{aligned} & \log V(x(t)) + (\hat{a^l_{ij}})\int_0^t V(x(s))ds \\ \le\ & \log V(x(0)) + [(\breve{r^u_i}) + (\breve{b^u_i})]t + \int_0^t \frac{\sum\limits_{i=1}^n \sigma_i(s)x_i(s)dB_i(s)}{\sum\limits_{i=1}^n x_i(s)}. \qquad (39)\end{aligned}$$

However, it is straightforward to show by the strong law of large numbers of martingales that

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t \frac{\sum_{i=1}^n \sigma_i(s)x_i(s)dB_i(s)}{\sum_{i=1}^n x_i(s)} = 0 \text{ a.s.}$$

We can therefore divide both sides of (39) by t and then let $t\to\infty$ to obtain

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^t V(x(s))ds \le \frac{(\breve{r_i^u})+(\breve{b_i^u})}{(\hat{a_{ij}^l})} \text{ a.s.} \tag{40}$$

which implies the required assertion (33).

On the other hand, we observe from (37) that

$$d[\log V(x(t))] \ge [\frac{(\hat{r_i^l})}{n} - (\breve{a_{ij}^u})V(x(t))]dt + \frac{\sum_{i=1}^n \sigma_i(t)x_i(t)dB_i(t)}{\sum_{i=1}^n x_i(t)}. \tag{41}$$

Hence

$$\begin{aligned}\log V(x(t)) \quad &+ \quad (\breve{a_{ij}^u})\int_0^t V(x(s))ds \ge \\ \log V(x(0)) \quad &+ \quad \frac{(\hat{r_i^l})}{n}t + \int_0^t \frac{\sum_{i=1}^n \sigma_i(s)x_i(s)dB_i(s)}{\sum_{i=1}^n x_i(s)}.\end{aligned} \tag{42}$$

So we have

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t V(x(s))ds \ge \frac{(\hat{r_i^l})}{n(\breve{a_{ij}^u})} \text{ a.s.} \tag{43}$$

Noting that $V(x(t)) \le n^{\frac{1}{2}}|x(t)|$, we obtain the other required assertion (34). □

As special case, let us consider the autonomous n-species competitive system with random perturbation

$$dx_i(t) = x_i(t)\left[\left(b_i - \sum_{i=1}^n a_{ij}x_j(t)\right)dt + \sigma_i dB_i(t)\right],\ i = 1, 2, \cdots, n, \tag{44}$$

where $b_i,\ a_{ij}$ are nonnegative constants, $i,\ j = 1,\ \cdots,\ n$.

Corollary 4. *Assume that for each $1 \le i, j \le n$, $a_{ij} > 0$, $b_i - \frac{1}{2}\sigma_i^2 > 0$. Then for any initial value $x(0) \in R_+^n$, the solution $x(t)$ to equation (44) has the following property*

$$\frac{(\hat{r_i})}{n^{\frac{3}{2}}(\breve{a_{ij}})} \le \liminf_{t\to+\infty}\frac{1}{t}\int_0^t |x(s)|ds \le \limsup_{t\to+\infty}\frac{1}{t}\int_0^t |x(s)|ds \le \frac{(\breve{r_i})+(\breve{b_i})}{(\hat{a_{ij}})} \quad a.s. \tag{45}$$

6. **Global Attractivity.** In this section, we turn to establishing sufficient criteria for the global attractivity of stochastic equation (2).

Definition 6.1. Let $x(t),\ y(t)$ be two arbitrary solutions of equation (2) with initial values $x(0),\ y(0) \in R_+^n$ respectively. If

$$\lim_{t \to +\infty} |x(t) - y(t)| = 0 \text{ a.s.}$$

then we say equation (2) is globally attractive.

Assumption 5. *$(a_{ii} - \sum_{j=1, j \neq i}^{n} a_{ji})^l > 0$ for all $1 \leq i \leq n$.*

Theorem 6.2. *Under Assumption 5, equation (2) is globally attractive.*

Proof. The proof is rather technical so we divide it into two steps.
Step 1. Let $x(t),\ y(t)$ be two arbitrary solutions of equation (2) with initial values $x(0),\ y(0) \in R_+^n$. By the Itô formula, we have

$$\begin{aligned} d \log x_i(t) &= \left(b_i(t) - \frac{\sigma_i^2(t)}{2} - \sum_{j=1}^{n} a_{ij}(t) x_j(t) \right) dt + \sigma_i(t) dB_i(t), \\ d \log y_i(t) &= \left(b_i(t) - \frac{\sigma_i^2(t)}{2} - \sum_{j=1}^{n} a_{ij}(t) y_j(t) \right) dt + \sigma_i(t) dB_i(t). \end{aligned}$$

Then,

$$d(\log x_i(t) - \log y_i(t)) = -\sum_{j=1}^{n} a_{ij}(t)(x_j(t) - y_j(t))dt. \tag{46}$$

Consider a Lyapunov function $\bar{V}(t)$ defined by

$$\bar{V}(t) = \sum_{i=1}^{n} |\log x_i(t) - \log y_i(t)|,\ t \geq 0.$$

A direct calculation of the right differential $d^+\bar{V}(t)$ of $\bar{V}(t)$ along the ordinary differential equation (46) leads to

$$\begin{aligned}
d^+\bar{V}(t) &= \sum_{i=1}^{n} \operatorname{sgn}(x_i(t)-y_i(t))d(\log x_i(t)-\log y_i(t)) \\
&= -\sum_{i=1}^{n} \operatorname{sgn}(x_i(t)-y_i(t))\sum_{j=1}^{n} a_{ij}(t)(x_j(t)-y_j(t))dt \\
&\le -\sum_{i=1}^{n} a_{ii}(t)|x_i(t)-y_i(t)|dt + \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} a_{ij}(t)|x_j(t)-y_j(t)|dt \\
&= -\sum_{i=1}^{n} a_{ii}(t)|x_i(t)-y_i(t)|dt + \sum_{j=1}^{n}\sum_{i=1,i\neq j}^{n} a_{ji}(t)|x_i(t)-y_i(t)|dt \\
&= -\sum_{i=1}^{n} a_{ii}(t)|x_i(t)-y_i(t)|dt + \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} a_{ji}(t)|x_i(t)-y_i(t)|dt \\
&\le -\sum_{i=1}^{n}(a_{ii} - \sum_{j=1,j\neq i}^{n} a_{ji})^l|x_i(t)-y_i(t)|dt \\
&\le -\varphi\sum_{i=1}^{n}|x_i(t)-y_i(t)|dt, \qquad (47)
\end{aligned}$$

where $\varphi = \min\limits_{1\le i\le n}(a_{ii} - \sum\limits_{j=1,j\neq i}^{n} a_{ji})^l > 0$. Integrating (6.2) from 0 to t, we have

$$\bar{V}(t) + \varphi\int_0^t \sum_{i=1}^{n}|x_i(s)-y_i(s)|ds \le \bar{V}(0) < \infty.$$

Let $t\to\infty$, we obtain that

$$\int_0^\infty |x(s)-y(s)|ds \le \int_0^\infty \sum_{i=1}^{n}|x_i(s)-y_i(s)|ds \le \frac{\bar{V}(0)}{\varphi} < \infty \text{ a.s.} \qquad (48)$$

Moreover, we also have

$$E\int_0^\infty |x(s)-y(s)|ds < \infty. \qquad (49)$$

Step 2. Set $u(t) = x(t)-y(t)$. Clearly, $u \in C(R_+, R)$ a.s. It is straightforward to see from (48) that

$$\liminf_{t\to\infty}|u(t)| = 0 \text{ a.s.} \qquad (50)$$

We now claim that

$$\lim_{t\to\infty}|u(t)| = 0 \text{ a.s.} \qquad (51)$$

If this is false, then

$$P\left\{\limsup_{t\to\infty}|u(t)| > 0\right\} > 0.$$

Hence there is a (fixed) number $\epsilon > 0$ such that

$$P(\Omega_1) \ge 2\epsilon, \qquad (52)$$

where

$$\Omega_1 = \left\{ \limsup_{t\to\infty} |u(t)| > 2\epsilon \right\}.$$

Let us now define a sequence of stopping times,

$$\sigma_1 = \inf\{t \ge 0 : \ |u(t)| \ge 2\epsilon\}, \qquad \sigma_{2k} = \inf\{t \ge \sigma_{2k-1} : \ |u(t)| \le \epsilon\},$$
$$\sigma_{2k+1} = \inf\{t \ge \sigma_{2k} : \ |u(t)| \ge 2\epsilon\}, \qquad k = 1, 2, \dots$$

Note from (50) and the definition of Ω_1 that

$$\sigma_k < \infty, \text{ for } \forall k \ge 1 \text{ whenever } \ \omega \in \ \Omega_1. \tag{53}$$

By (49), we compute

$$\begin{aligned}
\infty &> E\int_0^\infty |u(s)|ds \\
&\ge \sum_{k=1}^\infty E\left[I_{\{\sigma_{2k-1}<\infty,\ \sigma_{2k}<\infty\}} \int_{\sigma_{2k-1}}^{\sigma_{2k}} |u(s)|ds \right] \\
&\ge \epsilon \sum_{k=1}^\infty E\left[I_{\{\sigma_{2k-1}<\infty\}} (\sigma_{2k-1} - \sigma_{2k}) \right],
\end{aligned} \tag{54}$$

where I_A is the indicator function of set A and we have noted from (50) that $\sigma_{2k} < \infty$ whenever $\sigma_{2k-1} < \infty$.

On the other hand, we rewrite equation(2) as

$$x_i(t) = x_i(0) + \int_0^t f_i(s, x(s))ds + \int_0^t g_i(s, x(s))dB_i(s), \tag{55}$$

where

$$\begin{aligned}
f_i(s, x(s)) &= x_i(s)(b_i(s) - \textstyle\sum_{j=1}^n a_{ij}(s)x_j(s)); \\
g_i(s, x(s)) &= \sigma_i(s)x_i(s).
\end{aligned}$$

Then,

$$\begin{aligned}
E(|f_i(s, x(s))|^2) &= E(x_i^2(t)|b_i(s) - \sum_{j=1}^n a_{ij}(s)x_j(s)|^2) \\
&\le \frac{1}{2}E(x_i^4(s)) + \frac{1}{2}E[(b_i(s) - \sum_{j=1}^n a_{ij}(s)x_j(s))^4] \\
&\le \frac{1}{2}E(x_i^4(s)) + \frac{1}{2}(n+1)^3 E(b_i^4(s) + \sum_{j=1}^n a_{ij}^4(s)x_j^4(s)) \\
&\le \frac{1}{2}L_i(4, x(0)) + \frac{1}{2}(n+1)^3 (b_i^4(s) + \sum_{j=1}^n a_{ij}^4(s)L_j(4, x(0))) \\
&\le \frac{1}{2}L_i(4, x(0)) + \frac{1}{2}(n+1)^3 [(b_i^u)^4 + \sum_{j=1}^n (a_{ij}^u)^4 L_j(4, x(0))] \\
&=: F_i(2, x(0)),
\end{aligned} \tag{56}$$

and

$$\begin{aligned}
E(|g_i(s, x(s))|^2) &= E(|\sigma_i(s)|^2 x_i^2(s)) = |\sigma_i(s)|^2 E(x_i^2(s)) \\
&\le (\sigma_i^u)^2 L_i(2, x(0)) =: G_i(2, x(0)),
\end{aligned} \tag{57}$$

where the positive constant $L_i(p, x(0))$ denotes the upper boundary of $E(x_i^p(t))$ with initial value $x(0) \in R_+^n$ for any positive constant p. Using the Hölder inequality and the moment inequality of stochastic integrals (see [24, 27]), we compute

$$
\begin{aligned}
& E\left[I_{\{\sigma_{2k-1}<\infty\}} \sup_{0\le t\le T} |x_i(\sigma_{2k-1}+t) - x_i(\sigma_{2k-1})|^2\right] \\
\le\ & 2E\left[I_{\{\sigma_{2k-1}<\infty\}} \sup_{0\le t\le T} \left|\int_{\sigma_{2k-1}}^{\sigma_{2k-1}+t} f_i(s,x(s))ds\right|^2\right] \\
& +2E\left[I_{\{\sigma_{2k-1}<\infty\}} \sup_{0\le t\le T} \left|\int_{\sigma_{2k-1}}^{\sigma_{2k-1}+t} g_i(s,x(s))dB_i(s)\right|^2\right] \\
\le\ & 2TE\left[I_{\{\sigma_{2k-1}<\infty\}} \int_{\sigma_{2k-1}}^{\sigma_{2k-1}+T} |f_i(s,x(s))|^2 ds\right] \\
& +8E\left[I_{\{\sigma_{2k-1}<\infty\}} \int_{\sigma_{2k-1}}^{\sigma_{2k-1}+T} |g_i(s,x(s))|^2 ds\right] \\
\le\ & 2(T+4)T[F_i(2,x(0)) + G_i(2,x(0))],
\end{aligned}
$$

then we have

$$
\begin{aligned}
& E\left[I_{\{\sigma_{2k-1}<\infty\}} \sup_{0\le t\le T} |x(\sigma_{2k-1}+t) - x(\sigma_{2k-1})|^2\right] \\
=\ & E\left[I_{\{\sigma_{2k-1}<\infty\}} \sup_{0\le t\le T} \sum_{i=1}^n |x_i(\sigma_{2k-1}+t) - x_i(\sigma_{2k-1})|^2\right] \\
\le\ & \sum_{i=1}^n E\left[I_{\{\sigma_{2k-1}<\infty\}} \sup_{0\le t\le T} |x_i(\sigma_{2k-1}+t) - x_i(\sigma_{2k-1})|^2\right] \\
\le\ & 2(T+4)T \sum_{i=1}^n [F_i(2,x(0)) + G_i(2,x(0))] \qquad (58)
\end{aligned}
$$

In the same way, we have

$$
\begin{aligned}
& E\left[I_{\{\sigma_{2k-1}<\infty\}} \sup_{0\le t\le T} |y(\sigma_{2k-1}+t) - y(\sigma_{2k-1})|^2\right] \\
\le\ & 2(T+4)T \sum_{i=1}^n [F_i(2,y(0)) + G_i(2,y(0))]. \qquad (59)
\end{aligned}
$$

Let

$$F_i(2) = \max\{F_i(2,x(0)),\ F_i(2,y(0))\},\ G_i(2) = \max\{G_i(2,x(0)),\ G_i(2,y(0))\}.$$

We furthermore choose $T = T(\epsilon) > 0$ sufficiently small for

$$16(T+4)T \sum_{i=1}^n [F_i(2) + G_i(2)] \le \epsilon^3.$$

It then follows from (58) and (59) that

$$P(\{\sigma_{2k-1} < \infty\} \cap \Omega_k^1) \leq \frac{2(T+4)T\sum_{i=1}^{n}[F_i(2)+G_i(2)]}{\frac{\epsilon^2}{4}} \leq \frac{\epsilon}{2}, \tag{60}$$

$$P(\{\sigma_{2k-1} < \infty\} \cap \Omega_k^2) \leq \frac{2(T+4)T\sum_{i=1}^{n}[F_i(2)+G_i(2)]}{\frac{\epsilon^2}{4}} \leq \frac{\epsilon}{2}. \tag{61}$$

where

$$\Omega_k^1 = \left\{ \sup_{0\leq t\leq T} |x(\sigma_{2k-1}+t) - x(\sigma_{2k-1})| \geq \frac{\epsilon}{2} \right\},$$

$$\Omega_k^2 = \left\{ \sup_{0\leq t\leq T} |y(\sigma_{2k-1}+t) - y(\sigma_{2k-1})| \geq \frac{\epsilon}{2} \right\}.$$

It easy to see from (60) and (61) that

$$P(\{\sigma_{2k-1} < \infty\} \cap (\Omega_k^1 \cup \Omega_k^2)) \leq \epsilon. \tag{62}$$

Recalling (53), we further compute

$$\begin{aligned} & P(\{\sigma_{2k-1} < \infty\} \cap (\overline{\Omega_k^1} \cap \overline{\Omega_k^2})) \\ = \quad & P(\{\sigma_{2k-1} < \infty\}) - P(\{\sigma_{2k-1} < \infty\} \cap (\Omega_k^1 \cup \Omega_k^2)) \\ \geq \quad & 2\epsilon - \epsilon = \epsilon. \end{aligned}$$

Then we easily know that

$$\begin{aligned} & P(\{\sigma_{2k-1} < \infty\} \cap \left\{ \sup_{0\leq t\leq T} |u(\sigma_{2k-1}+t) - u(\sigma_{2k-1})| < \epsilon \right\}) \\ \geq \quad & P(\{\sigma_{2k-1} < \infty\} \cap (\overline{\Omega_k^1} \cap \overline{\Omega_k^2})) \qquad (63) \\ \geq \quad & \epsilon. \qquad (64) \end{aligned}$$

Set

$$\Omega_k^3 = \left\{ \sup_{0\leq t\leq T} |u(\sigma_{2k-1}+t) - u(\sigma_{2k-1})| < \epsilon \right\}.$$

Noting that

$$\sigma_{2k}(\omega) - \sigma_{2k-1}(\omega) \geq T, \text{ if } \omega \in \{\sigma_{2k-1} < \infty\} \cap \Omega_k^3,$$

we derive from (54) and (64) that

$$\begin{aligned} \infty \quad > \quad & \epsilon \sum_{k=1}^{\infty} E\left[I_{\{\sigma_{2k-1}<\infty\}}(\sigma_{2k-1} - \sigma_{2k}) \right] \\ \geq \quad & \epsilon \sum_{k=1}^{\infty} E\left[I_{\{\sigma_{2k-1}<\infty\}\cap\Omega_k^3}(\sigma_{2k-1} - \sigma_{2k}) \right] \\ \geq \quad & \epsilon T \sum_{k=1}^{\infty} P(\{\sigma_{2k-1} < \infty\} \cap \Omega_k^3) \\ \geq \quad & \epsilon T \sum_{k=1}^{\infty} \epsilon = \infty, \qquad (65) \end{aligned}$$

which is a contraction. So (51) must hold. □

Corollary 5. *If $a^l > 0$, equation (3) is globally attractive.*

This corollary is one of the main results in [11].

Appendix.

Appendix A. **Proof of Theorem 2.1.** Since the coefficients of the equation are locally Lipschitz continuous, for any given initial value $x(0) \in R^n_+$ there is an unique local solution $x(t)$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. To show this solution is global, we need to show that $\tau_e = \infty$ a.s. Let $k_0 > 0$ be sufficiently large for every component of $x(0)$ lying within the interval $[\frac{1}{k_0}, k_0]$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \in [0,\ \tau_e) : x_i(t) \notin (\frac{1}{k},\ k) \text{ for some } i = 1,\ \cdots,\ n\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $x(t) \in R^n_+$ a.s. for all $t \geq 0$. In other words, to complete the proof all we need to show is that $\tau_\infty = \infty$ a.s. If this statement is false, there is a pair of constants $T > 0$ and $\varepsilon \in (0,\ 1)$ such that

$$P\{\tau_\infty \leq T\} > \varepsilon.$$

Hence there is an integer $k_1 \geq k_0$ such that

$$P\{\tau_k \leq T\} \geq \varepsilon \text{ for all } k \geq k_1. \tag{66}$$

Define a C^2-function $V : R^n_+ \to R_+$ by

$$V(x) = \sum_{i=1}^n [x_i - 1 - \log(x_i)].$$

The nonnegativity of this function can be seen from

$$u - 1 - \log(u) \geq 0 \quad \text{on } u > 0.$$

If $x(t) \in R^n_+$, the Itô formula shows that

$$\begin{aligned}
&dV(x(t)) \\
&= \sum_{i=1}^n \left\{(1 - x_i^{-1})x_i \left[\left(b_i(t) - \sum_{j=1}^n a_{ij}(t)x_j\right) dt + \sigma_i(t)dB_i(t)\right] + 0.5\sigma_i^2(t)dt\right\} \\
&= \sum_{i=1}^n \left[(x_i - 1)\left(b_i(t) - \sum_{j=1}^n a_{ij}(t)x_j\right) + 0.5\sigma_i^2(t)\right] dt \\
&\quad + \sum_{i=1}^n (x_i - 1)\sigma_i(t)dB_i(t) \\
&=: F(x)dt + \sum_{i=1}^n (x_i - 1)\sigma_i(t)dB_i(t),
\end{aligned}$$

where we write $x(t) = x$. Compute

$$\begin{aligned}
&\sum_{i=1}^n (x_i - 1)\left(b_i(t) - \sum_{j=1}^n a_{ij}(t)x_j\right) \\
&= \sum_{i=1}^n b_i(t)(x_i - 1) + \sum_{i=1}^n (1 - x_i) \sum_{j=1}^n a_{ij}(t)x_j \\
&\leq (\breve{b_i^u}) \sum_{i=1}^n x_i + \sum_{i=1}^n (1 - x_i) \sum_{j=1}^n a_{ij}(t)x_j
\end{aligned}$$

Since for each $1 \le i \le n$, $(1 - x_i) \sum_{j=1}^{n} a_{ij}(t)x_j \le (\breve{a_{ij}^u}) \sum_{i=1}^{n} x_i$, we obtain that

$$\sum_{i=1}^{n}(x_i - 1)\left(b_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j\right) \le [(\breve{b_i^u}) + n(\breve{a_{ij}^u})] \sum_{i=1}^{n} x_i,$$

together with $u \le 2[u - 1 - \log(u)] + 2$ on $u > 0$, we know that

$$\begin{aligned} F(x) &\le [(\breve{b_i^u}) + n(\breve{a_{ij}^u})] \sum_{i=1}^{n} x_i + 0.5n(\breve{\sigma_i^u})^2 \\ &\le 2[(\breve{b_i^u}) + n(\breve{a_{ij}^u})][V(x) + n] + 0.5n(\breve{\sigma_i^u})^2 \\ &= 2[(\breve{b_i^u}) + n(\breve{a_{ij}^u})]V(x) + n[2(\breve{b_i^u}) + 2n(\breve{a_{ij}^u}) + 0.5(\breve{\sigma_i^u})^2] \\ &=: K_1^* V(x) + K_2^*, \end{aligned}$$

since $x(t) \in R_+^n$. We therefore obtain

$$\begin{aligned} &\int_0^{\tau_k \wedge T} dV(x(t)) \\ &\le \int_0^{\tau_k \wedge T} [K_1^* V(x(t)) + K_2^*]dt + \int_0^{\tau_k \wedge T} \sum_{i=1}^{n}(x_i(t) - 1)\sigma_i(t)dB_i(t), \end{aligned}$$

Whence taking expectations, yields

$$\begin{aligned} &EV(x(\tau_k \wedge T)) \\ &\le V(x(0)) + K_2^* E(\tau_k \wedge T) + K_1^* E \int_0^{\tau_k \wedge T} V(x(t))dt \\ &\le V(x(0)) + K_2^* T + K_1^* \int_0^{T} EV(x(\tau_k \wedge t))dt. \end{aligned}$$

The Gronwall inequality implies that

$$EV(x(\tau_k \wedge T)) \le [V(x(0)) + K_2^* T]e^{K_1^* T}. \tag{67}$$

Set $\Omega_k = \{\tau_k \le T\}$ for $k \ge k_1$ and by (66), $P(\Omega_k) \ge \varepsilon$. Note that for every $\omega \in \Omega_k$, there is some i such that $x_i(\tau_k,\ \omega)$ equals either k or $\frac{1}{k}$, and hence $V(x(\tau_k,\ \omega))$ is no less than either

$$k - 1 - \log(k)$$

or

$$\frac{1}{k} - 1 - \log(\frac{1}{k}) = \frac{1}{k} - 1 + \log(k).$$

Consequently,

$$V(x(\tau_k, \omega)) \ge [k - 1 - \log(k)\ ] \wedge [\log(k) - 1 + \frac{1}{k}\ ].$$

It then follows from (67) that

$$\begin{aligned} [V(x(0)) + K_2^* T]e^{K_1^* T} &\ge E[1_{\Omega_k}(\omega)V(x(\tau_k, \omega))] \\ &\ge \varepsilon([k - 1 - \log(k)] \wedge [\log(k) - 1 + \tfrac{1}{k}\ ]), \end{aligned}$$

where 1_{Ω_k} is the indicator function of Ω_k. Letting $k \to \infty$ leads to the contradiction

$$\infty > [V(x(0)) + K_2^* T]e^{K_1^* T} = \infty.$$

So we must have $\tau_\infty = \infty$ a.s. This completes the proof of Theorem 2.1.

Appendix B. **Proof of Lemma 3.4.** Define

$$V(x)=\sum_{i=1}^{n}x_i \quad \text{for } x\in R_+^n.$$

By the Itô formula we can show that

$$\begin{aligned} E\left(\sup_{t\le r\le t+1}V(x(r))\right) &\le EV(x(t))+(\breve{b_i^u})\int_t^{t+1}EV(x(s))ds \\ &+ E\left(\sup_{t\le r\le t+1}\int_t^r\sum_{i=1}^{n}\sigma_i(s)x_i(s)dB_i(s)\right). \end{aligned}$$

By Lemma 3.1, it is easy to see that

$$\limsup_{t\to\infty}EV(x(t))\le\sum_{i=1}^{n}K_i(1)=n^{-\frac{1}{2}}K(1). \tag{68}$$

and

$$\limsup_{t\to\infty}E\int_t^{t+1}|x(s)|^2ds\le K(2). \tag{69}$$

But, by the well-known Burkholder–Davis–Gundy inequality (see [24, 27]) and the Hölder inequality, we can show that

$$E\left(\sup_{t\le r\le t+1}\int_t^r\sum_{i=1}^{n}\sigma_i(s)x_i(s)dB_i(s)\right)\le 3(\breve{\sigma_i^u})\left[E\int_t^{t+1}|x(s)|^2ds\right]^{\frac{1}{2}}.$$

Therefore

$$\begin{aligned} &E\left(\sup_{t\le r\le t+1}V(x(r))\right) \\ &\le EV(x(t))+(\breve{b_i^u})\textstyle\int_t^{t+1}EV(x(s))ds+3(\breve{\sigma_i^u})\left[E\displaystyle\int_t^{t+1}|x(s)|^2ds\right]^{\frac{1}{2}}. \end{aligned}$$

This, together with (68) and (69), yields

$$\limsup_{t\to\infty}E\left(\sup_{t\le r\le t+1}V(x(r))\right)\le[1+(\breve{b_i^u})]n^{-\frac{1}{2}}K(1)+3(\breve{\sigma_i^u})[K(2)]^{\frac{1}{2}}.$$

Noting that $|x(t)|\le\sum_{i=1}^{n}x_i(t)$, we obtain assertion (6).

To prove the other assertion (7) we observe from (6) that there is a positive constant $\bar{K}$ such that

$$E\left(\sup_{k\le t\le k+1}|x(t)|\right)\le\bar{K},\ k=1,2,\ldots$$

Let $\epsilon>0$ be arbitrary. Then, by the well-known Chebyshev inequality, we have

$$P\left\{\sup_{k\le t\le k+1}|x(t)|>k^{1+\epsilon}\right\}\le\frac{\bar{K}}{k^{1+\epsilon}},\ k=1,2,\ldots$$

Applying the well-known Borel-Cantelli lemma (see e.g. [24]), we obtain that for almost all $\omega\in\Omega$

$$\sup_{k\le t\le k+1}|x(t)|\le k^{1+\epsilon} \tag{70}$$

holds for all but finitely many k. Hence, there exists a $k_0(\omega)$, for almost all $\omega \in \Omega$, for which (70) holds whenever $k \geq k_0$. Consequently, for almost all $\omega \in \Omega$, if $k \geq k_0$ and $k \leq t \leq k+1$,

$$\frac{\log(|x(t)|)}{\log t} \leq \frac{(1+\epsilon)\log k}{\log k} = 1+\epsilon.$$

Therefore

$$\limsup_{t\to\infty} \frac{\log(|x(t)|)}{\log t} \leq 1+\epsilon. \text{ a.s.}$$

Letting $\epsilon \to 0$ we obtain the desired assertion (7). The proof is therefore complete.

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