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ASYMPTOTIC PROPERTIES OF STOCHASTIC POPULATION DYNAMICS

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Abstract. In this paper we stochastically perturb the classical Lotka–Volterra model

$$\dot{x}(t) = \text{diag}(x_1(t), \cdots, x_n(t))[b + Ax(t)]$$

into the stochastic differential equation

$$dx(t) = \text{diag}(x_1(t), \cdots, x_n(t))[(b + Ax(t))dt + \beta dw(t)].$$

The main aim is to study the asymptotic properties of the solution. It is known (see e.g. [3, 20]) if the noise is too large then the population may become extinct with probability one. Our main aim here is to find out what happens if the noise is relatively small. In this paper we will establish some new asymptotic properties for the moments as well as for the sample paths of the solution. In particular, we will discuss the limit of the average in time of the sample paths.

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1 Introduction

The classical Lotka–Volterra model for n interacting species is described by the n-dimensional differential equation

$$\frac{dx(t)}{dt} = \text{diag}(x_1(t), \cdots, x_n(t))[b + Ax(t)], \tag{1.1}$$

where

$$x = (x_1, \cdots, x_n)^T, \quad b = (b_1, \cdots, b_n)^T, \quad A = (a_{ij})_{n\times n}.$$

There is an extensive literature concerned with the dynamics of this model and we here only mention Ahmad and Rao [1], Bereketoglu and Gyori [4], Freedman and Ruan [9], He and Gopalsamy [11], Kuang and Smith [15], Teng and Yu [24] among many others. In particular, the books by Gopalsamy [10], Kolmanovskii and Myshkis [13] as well as Kuang [14] are good references in this area.

On the other hand, population systems are often subject to environmental noise. It is therefore useful to reveal how the noise affects the population systems. As a matter of fact, stochastic population systems have recently been studied by many authors, for example, [2, 3, 6, 7, 8, 20, 21, 23]. In particular, Mao, Marion and Renshaw [21] revealed that the environmental noise can suppress a potential population explosion while Mao [20] showed that different structures of environmental noise may have different effects on the population systems.

In this paper we consider the simple situation of the parameter perturbation. Recall that the parameter b_i represents the intrinsic growth rate of species i. In practice we usually estimate it by an average value plus an error term. If we still use b_i to denote the average growth rate, then the intrinsic growth rate becomes

$$b_i + \text{error}_i.$$

Let us consider a small subsequent time interval dt, during which $x_i(t)$ changes to $x_i(t) + dx_i(t)$. (We use the notation $d\cdot$ for the small change in any quantity over this time interval when we intend to consider it as an infinitesimal change.) Accordingly, equation (1.1) becomes

$$\frac{dx_i(t)}{dt} = (b_i + \text{error}_i)x(t) + \sum_{j=1}^{n} a_{ij}x_i x_j$$

for $1 \le i \le n$, that is

$$dx_i(t) = x_i(t)\big(b_i + \sum_{j=1}^{n} a_{ij}x_j\big)dt + x_i(t)\,\text{error}_i\,dt \tag{1.2}$$

According to the well-known central limit theorem, the error term $\text{error}_i\,dt$ may be approximated by a normal distribution with mean zero and variance $\beta_i^2 dt$. In terms of mathematics,

$$\text{error}_i\,dt \sim N(0, \beta_i^2 dt),$$

which can be written as

$$\text{error}_i\,dt \sim \beta_i dw(t)$$

where $dw(t) = w(t + dt) - w(t)$ is the increment of a Brownian motion that follows $N(0, dt)$. Hence equation (1.2) becomes the Itô stochastic differential equation

$$dx_i(t) = x_i(t)\big(b_i + \sum_{j=1}^{n} a_{ij}x_j\big)dt + x_i(t)\beta_i dw(t),$$

that is, in the matrix form,

$$dx(t) = \text{diag}(x_1(t), \cdots, x_n(t))[(b + Ax(t))dt + \beta dw(t)], \tag{1.3}$$

where $\beta = (\beta_1, \cdots, \beta_n)^T$ that will be called the intensities of noise. For more biological motivation on this type of modelling in population dynamics we refer the reader to Gard [6, 7, 8]. We should mention that such an idea of stochastic modelling has also been used widely in mathemtical finance, for example, in the Nobel prize winning model, i.e. the geometric Brownian motion. Since equation (1.3) describes a stochastic population dynamics, it is critical for the solution to remain positive and not to explode to infinity in a finite time. Sufficient conditions for these properties is one of the important topics in the study of stochastic population systems (see e.g. [3, 8, 20, 23]). However, the main aim of this paper is to discuss asymptotic properties of the solution.

It is known (see e.g. [3, 20]) if the noise is too large then the population may become extinct. However, it is interesting to find out what happens if the noise is relatively small. In this paper we will establish some new asymptotic properties for the moments as well as for the sample paths of the solution. For example, we will show that if $-A$ is a non-singular M-matrix and $b_i > \frac{1}{2}\beta_i^2$ $(1 \le i \le n)$, then the average in time of the solution, $\frac{1}{t}\int_0^t x(s)ds$, has its finite limit with probability one and the limit is a solution to a simple linear equation.

2 Preliminaries

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions (i.e. it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $w(t)$ denote a scalar Brownian motion defined on this probability space. We also denote by $\mathbb{R}_+^n$ the positive cone in $\mathbb{R}^n$, that is $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_i > 0 \text{ for all } 1 \le i \le n\}$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$ whilst its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$.

In this paper we will use a lot of quadratic functions of the form $x^T A x$ for the state $x \in \mathbb{R}_+^n$ only. Therefore, for a symmetric $n \times n$ matrix $A = (a_{ij})_{n\times n}$, we recall the following definition

$$\lambda_{\max}^+(A) = \sup_{x\in\mathbb{R}_+^n, |x|=1} x^T A x,$$

which was introduced by Bahar and Mao [3]. Let us emphasise that this is different from the largest eigenvalue $\lambda_{\max}(A)$ of the matrix A. To see this more clearly, let us recall the nice property of the largest eigenvalue:

$$\lambda_{\max}(A) = \sup_{x\in\mathbb{R}^n, |x|=1} x^T A x.$$

It is therefore clear that we always have

$$\lambda_{\max}^+(A) \le \lambda_{\max}(A).$$

In many situations we even have $\lambda^+_{\max}(A) < \lambda_{\max}(A)$. For example, for

$$A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},$$

we have $\lambda^+_{\max}(A) = -1 < \lambda_{\max}(A) = 0$. On the other hand, $\lambda^+_{\max}(A)$ does have many similar properties as $\lambda_{\max}(A)$ has. For example, it follows straightforward from the definition that

$$x^T Ax \le \lambda^+_{\max}(A)|x|^2 \quad \forall x \in \mathbb{R}^n_+$$

and

$$\lambda^+_{\max}(A) \le \|A\|.$$

Moreover

$$\lambda^+_{\max}(A + B) \le \lambda^+_{\max}(A) + \lambda^+_{\max}(B)$$

if B is another symmetric $n \times n$ matrix. For more properties of $\lambda^+_{\max}(A)$ please see [3].

In this paper we will consider the stochastic population system (1.3) with initial value $x(0) \in \mathbb{R}^n_+$. As the ith state $x_i(t)$ of equation (1.3) is the size of the ith species in the system, it should be nonnegative. Moreover, in order for a stochastic differential equation to have a unique global (i.e. no explosion in a finite time) solution for any given initial value, the coefficients of the equation are generally required to satisfy the linear growth condition and local Lipschitz condition (see e.g. [16, 17, 18, 19, 25]). However, the coefficients of equation (1.3) do not satisfy the linear growth condition, though they are locally Lipschitz continuous, so the solution of equation (1.3) may explode at a finite time. It is therefore useful to establish some conditions under which the solution of equation (1.3) is not only positive but will also not explode to infinity at any finite time. The research in this direction is still a hot topic but we will not discuss it here. Instead, we cite a theorem from [3] for the use of this paper.

Theorem 2.1 *Assume that there are positive numbers $c_1, \cdots, c_n$ such that*

$$\lambda^+_{\max}(\bar{C}A + A^T\bar{C}) \le 0, \tag{2.1}$$

where $\bar{C} = \mathrm{diag}(c_1, \cdots, c_n)$. Then for any given initial value $x(0) \in \mathbb{R}^n_+$, there is a unique solution $x(t)$ to equation (1.3) on $t \ge 0$ and the solution will remain in $\mathbb{R}^n_+$ with probability 1, namely $x(t) \in \mathbb{R}^n_+$ for all $t \ge 0$ almost surely.

The theorem cited above follows from [3, Corollary 2.3] which is established for the more general stochastic delay population system

$$dx(t) = \mathrm{diag}(x_1(t), \cdots, x_n(t))[(b + Ax(t) + Bx(t-\tau))dt + \beta dw(t)]. \tag{2.2}$$

By setting $B = 0$ this equation becomes system (1.3) and hence Theorem 2.1 follows. The reader may find other alternative conditions in e.g. [3, 20] but we will mainly use condition (2.1) in this paper.

Bahar and Mao [3] also show that if the noise is sufficiently large, the solution of equation (1.3) will become extinct with probability one. To be more precise, we state another result which follows from [3, Theorem 4.1] by setting $B = 0$ in equation (2.2).

Theorem 2.2 *Let (2.1) hold. Assume moreover that the noise intensities* β_i *are sufficiently large in the sense that*

$$\beta_i \beta_j > b_i + b_j, \quad 1 \le i, j \le n. \tag{2.3}$$

Then for any given initial value $x(0) \in \mathbb{R}^n_+$*, the solution* $x(t)$ *of equation (1.3) has the property that*

$$\limsup_{t \to \infty} \frac{1}{t} \log(|x(t)|) \le -\frac{\varphi}{2} \quad a.s. \tag{2.4}$$

where

$$\varphi = \min_{1 \le i, j \le n} (\beta_i \beta_j - b_i - b_j) > 0.$$

That is, the population will become extinct exponentially with probability one.

This theorem reveals the important fact that the environmental noise may make the population extinct. For example, consider the scalar Lotka–Volterra model

$$\frac{dx(t)}{dt} = x(t)[\mu - \alpha x(t)]. \tag{2.5}$$

It is well known that if $\alpha > 0$ and $\mu > 0$, then for any $x(0) > 0$, its solution $x(t)$ obeys

$$\lim_{t \to \infty} x(t) = \frac{\mu}{\alpha}.$$

However, consider its associated stochastic Lotka–Volterra model

$$dx(t) = x(t)\Big([\mu - \alpha x(t)]dt + \sigma dw(t)\Big), \tag{2.6}$$

where $\sigma > 0$. By Theorem 2.2, if $\sigma^2 > 2\mu$, then the solution to this stochastic equation will become extinct with probability one, namely

$$\lim_{t \to \infty} x(t) = 0 \quad a.s.$$

The interesting question is: what happens if the noise is not so strong? Our main aim in this paper is to study the asymptotic properties of the solution of equation (1.3) when the noise is relatively small.

3 Asymptotic Properties

Let us begin with the following theorem.

Theorem 3.1 *Assume that there are positive numbers $c_1, \cdots, c_n$ such that*

$$-\lambda := \lambda^+_{\max}(\bar{C}A + A^T\bar{C}) < 0, \tag{3.1}$$

where $\bar{C} = \text{diag}(c_1, \cdots, c_n)$. Then for any given initial value $x(0) \in \mathbb{R}^n_+$, the solution $x(t)$ of equation (1.3) has the properties that

$$\limsup_{t\to\infty} \int_t^{t+1} \mathbb{E}|x(u)|^2 du \le \frac{4|C||\bar{C}b|}{\lambda^2}\Big(1 + \frac{|\bar{C}b|}{\hat{c}}\Big), \tag{3.2}$$

$$\begin{aligned} \limsup_{t\to\infty} \mathbb{E}\Big(\sup_{t\le u\le t+1} |x(u)|\Big) &\le \frac{2|C||\bar{C}b|}{\hat{c}}\Big(1 + \frac{|\bar{C}b|}{\hat{c}}\Big) \\ &+ \frac{6|\bar{C}\beta|}{\hat{c}}\sqrt{\frac{|C||\bar{C}b|}{\lambda^2}\Big(1 + \frac{|\bar{C}b|}{\hat{c}}\Big)}\ , \end{aligned} \tag{3.3}$$

and

$$\limsup_{t\to\infty} \frac{\log(|x(t)|)}{\log t} \le 1 \quad a.s. \tag{3.4}$$

where $\hat{c} = \min_{1\le i\le n} c_i$.

Proof. By Theorem 2.1, the solution $x(t)$ will remain in $\mathbb{R}^n_+$ for all $t \ge 0$ with probability 1. Set $C = (c_1, \cdots, c_n)^T \in \mathbb{R}^n_+$ and define

$$V(x) = C^T x = \sum_{i=1}^n c_i x_i \quad \text{for } x \in \mathbb{R}^n_+.$$

By the Itô formula, we have

$$dV(x(t)) = x^T(t)\bar{C}\big[(b + Ax(t))dt + \beta dw(t)\big]. \tag{3.5}$$

By condition (3.1),

$$x^T(t)\bar{C}Ax(t) = \frac{1}{2}x^T(t)(\bar{C}A + A^T\bar{C})x(t) \le -\tfrac{1}{2}\lambda|x(t)|^2.$$

It then follows from (3.5) that

$$dV(x(t)) \le \Big(|\bar{C}b||x(t)| - \tfrac{1}{2}\lambda|x(t)|^2\Big)dt + x^T(t)\bar{C}\beta dw(t). \tag{3.6}$$

Let $\gamma > 0$ be arbitrary. By the Itô formula once again, we have

$$\begin{aligned} d[e^{\gamma t}V(x(t))] &= e^{\gamma t}[\gamma V(x(t))dt + dV(x(t))] \\ &\le e^{\gamma t}\Big[(\gamma|C| + |\bar{C}b|)|x(t)| - \tfrac{1}{2}\lambda|x(t)|^2\Big]dt + e^{\gamma t}x^T(t)\bar{C}\beta dw(t). \end{aligned}$$

But

$$(\gamma|C| + |\bar{C}b|)|x(t)| - \tfrac{1}{2}\lambda|x(t)|^2 \le \frac{(\gamma|C| + |\bar{C}b|)^2}{2\lambda}.$$

So

$$d[e^{\gamma t}V(x(t))] \le \frac{e^{\gamma t}(\gamma|C| + |\bar{C}b|)^2}{2\lambda} + e^{\gamma t}x^T(t)\bar{C}\beta dw(t).$$

This implies

$$e^{\gamma t}\mathbb{E}V(x(t)) \le V(x(0)) + \frac{(e^{\gamma t}-1)(\gamma|C| + |\bar{C}b|)^2}{2\lambda\gamma}.$$

Hence

$$\limsup_{t\to\infty} \mathbb{E}V(x(t)) \le \frac{(\gamma|C| + |\bar{C}b|)^2}{2\lambda\gamma}.$$

Choosing $\gamma = |\bar{C}b|/|C|$, we obtain that

$$\limsup_{t\to\infty} \mathbb{E}V(x(t)) \le \frac{2|C||\bar{C}b|}{\lambda}. \tag{3.7}$$

Note that

$$|x(t)| \le \sum_{i=1}^n x_i(t) \le \frac{V(x(t))}{\hat{c}}. \tag{3.8}$$

Consequently

$$\limsup_{t\to\infty} \mathbb{E}|x(t)| \le \frac{2|C||\bar{C}b|}{\lambda\hat{c}}. \tag{3.9}$$

On the other hand, it follows from (3.6) that

$$0 \le \mathbb{E}V(x(t)) + |\bar{C}b|\int_t^{t+1} \mathbb{E}|x(u)|du - \tfrac{1}{2}\lambda\mathbb{E}\int_t^{t+1} |x(u)|^2 du.$$

This, together with (3.7) and (3.9), implies that

$$\limsup_{t\to\infty} \mathbb{E}\int_t^{t+1} |x(u)|^2 du \le \frac{4|C||\bar{C}b|}{\lambda^2}\Big(1 + \frac{|\bar{C}b|}{\hat{c}}\Big). \tag{3.10}$$

By the well-known Fubini theorem, we obtain the required assertion (3.2).

Moreover, we can also derive from (3.6) that

$$\begin{aligned}\mathbb{E}\Big(\sup_{t\le u\le t+1} V(x(u))\Big) \quad &\le \quad \mathbb{E}V(x(t)) + |\bar{C}b|\int_t^{t+1} \mathbb{E}|x(u)|du \\ &+ \quad \mathbb{E}\Big(\sup_{t\le u\le t+1}\int_t^u x^T(r)\bar{C}\beta dw(r)\Big).\end{aligned}$$

But, by the well-known Burkholder–Davis–Gundy inequality and the Hölder inequality, we derive that

$$\begin{aligned}\mathbb{E}\Big(\sup_{t\le u\le t+1}\int_t^u x^T(r)\bar{C}\beta dw(r)\Big) &\le 3\mathbb{E}\Big(\int_t^{t+1}|x^T(u)\bar{C}\beta|^2du\Big)^{\frac{1}{2}}\\ &\le 3|\bar{C}\beta|\Big(\mathbb{E}\int_t^{t+1}|x(u)|^2du\Big)^{\frac{1}{2}}.\end{aligned}$$

Therefore

$$\begin{aligned}\mathbb{E}\Big(\sup_{t\le u\le t+1}V(x(u))\Big) &\le \mathbb{E}V(x(t))+|\bar{C}b|\int_t^{t+1}\mathbb{E}|x(u)|du\\ &+ 3|\bar{C}\beta|\Big(\mathbb{E}\int_t^{t+1}|x(u)|^2du\Big)^{\frac{1}{2}}.\end{aligned}$$

This, together with (3.7), (3.9) and (3.10), yields

$$\begin{aligned}\limsup_{t\to\infty}\mathbb{E}\Big(\sup_{t\le u\le t+1}V(x(u))\Big) &\le \frac{2|C||\bar{C}b|}{\lambda}\Big(1+\frac{|\bar{C}b|}{\hat{c}}\Big)\\ &+ 6|\bar{C}\beta|\sqrt{\frac{|C||\bar{C}b|}{\lambda^2}\Big(1+\frac{|\bar{C}b|}{\hat{c}}\Big)}.\end{aligned}$$

Recalling (3.8) we obtain the another assertion (3.3).

To prove assertion (3.4) we observe from (3.3) that there is a positive constant K such that

$$\mathbb{E}\Big(\sup_{k\le u\le k+1}|x(u)|\Big)\le K,\quad k=1,2,\cdots.$$

Let $\varepsilon>0$ be arbitrary. Then, by the well-known Chebyshev inequality, we have

$$\mathbb{P}\Big\{\sup_{k\le t\le k+1}|x(t)|>k^{1+\varepsilon}\Big\}\le\frac{K}{k^{1+\varepsilon}},\quad k=1,2,\cdots.$$

Applying the well-known Borel–Cantelli lemma (see e.g. [19]), we obtain that for almost all $\omega\in\Omega$,

$$\sup_{k\le t\le k+1}|x(t)|\le k^{1+\varepsilon} \tag{3.11}$$

holds for all but finitely many k. Hence, there exists a $k_0(\omega)$, for almost all $\omega\in\Omega$, for which (3.11) holds whenever $k\ge k_0$. Consequently, for almost all $\omega\in\Omega$, if $k\ge k_0$ and $k\le t\le k+1$,

$$\frac{\log(|x(t)|)}{\log t}\le\frac{(1+\varepsilon)\log k}{\log k}=1+\varepsilon.$$

Therefore

$$\limsup_{t\to\infty}\frac{\log(|x(t)|)}{\log t}\le 1+\varepsilon\quad a.s.$$

Letting $\varepsilon \to 0$ we obtain the desired assertion (3.4). The proof is therefore complete. □

It is straightforward to see from assertion (3.2) that

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}|x(u)|^2 du \le \frac{4|C||\bar{C}b|}{\lambda^2}\Big(1+\frac{|\bar{C}b|}{\hat{c}}\Big).$$

This means the average in time of the second moment of the solution is bounded. Moreover, assertion (3.4) shows that for any $\varepsilon > 0$, there is a positive random variable T_ε such that, with probability one,

$$|x(t)| \le t^{1+\varepsilon} \quad \forall t \ge T_\varepsilon.$$

In other words, with probability one, the solution will not grow faster than $t^{1+\varepsilon}$. In the following theorem, we will show that under an additional condition, the solution will not decay faster than $t^{-(\theta+\varepsilon)}$, where θ will be specified precisely.

Theorem 3.2 *Assume that condition (2.1) holds. Assume also that*

$$\hat{b} := \min_{1\le i\le n} b_i > \frac{1}{2}\max_{1\le i\le n}\beta_i^2 := \frac{1}{2}\check{\beta}^2. \tag{3.12}$$

Then for any given initial value $x(0) \in \mathbb{R}_+^n$, the solution $x(t)$ of equation (1.3) has the property that

$$\liminf_{t\to\infty} \frac{\log(|x(t)|)}{\log t} \ge -\frac{\check{\beta}^2}{2\hat{b}-\check{\beta}^2} \quad a.s. \tag{3.13}$$

Proof. By Theorem 2.1, the solution $x(t)$ will remain in $\mathbb{R}_+^n$ for all $t \ge 0$ with probability 1. Let $V : \mathbb{R}_+^n \to (0,\infty)$ be the same as defined in the proof of Theorem 3.1 and define

$$y(t) = \frac{1}{V(x(t))} \quad \text{on } t \ge 0.$$

By the Itô formula, we derive from (3.5) that

$$\begin{aligned} dy(t) &= -y^2(t)dV(x(t)) + y^3(t)(x^T(t)\bar{C}\beta)^2 dt \\ &= \big[-y^2(t)x^T(t)\bar{C}(b+Ax(t)) + y^3(t)(x^T(t)\bar{C}\beta)^2\big]dt \\ &- y^2(t)x^T(t)\bar{C}\beta dw(t). \end{aligned} \tag{3.14}$$

Choose any θ such that

$$0 < \theta < \frac{2\hat{b}}{\check{\beta}^2} - 1. \tag{3.15}$$

Define

$$z(t) = 1 + y(t) \quad \text{on } t \ge 0.$$

Applying the Itô formula again we have

$$\begin{aligned} dz^\theta(t) &= \theta z^{\theta-1}(t)dy(t) - \tfrac{1}{2}\theta(1-\theta)z^{\theta-2}(t)y^4(t)(x^T(t)\bar{C}\beta)^2 dt \\ &= \theta z^{\theta-2}(t)\Big(z(t)\big[-y^2(t)x^T(t)\bar{C}(b+Ax(t)) + y^3(t)(x^T(t)\bar{C}\beta)^2\big] \\ &\quad -\tfrac{1}{2}(1-\theta)y^4(t)(x^T(t)\bar{C}\beta)^2\Big)dt \\ &- \theta z^{\theta-1}(t)y^2(t)x^T(t)\bar{C}\beta dw(t). \end{aligned} \tag{3.16}$$

Dropping t from $x(t)$ etc. we compute that

$$\begin{aligned} & z\big[-y^2x^T\bar{C}(b+Ax) + y^3(x^T\bar{C}\beta)^2\big] - \tfrac{1}{2}(1-\theta)y^4(x^T\bar{C}\beta)^2 \\ & = -y^2x^T\bar{C}b - y^2x^T\bar{C}Ax - y^3x^T\bar{C}b - y^3x^T\bar{C}Ax \\ & \quad + y^3(x^T\bar{C}\beta)^2 + \tfrac{1}{2}(1+\theta)y^4(x^T\bar{C}\beta)^2 \\ & \le -\frac{x^T\bar{C}Ax}{V^2(x)} + \frac{(x^T\bar{C}\beta)^2 - x^T\bar{C}Ax}{V^2(x)}y - \Big(\frac{x^T\bar{C}b}{V(x)} - \tfrac{1}{2}(1+\theta)\frac{(x^T\bar{C}\beta)^2}{V^2(x)}\Big)y^2. \end{aligned}$$

It is easy to see that for all $x \in \mathbb{R}^n_+$,

$$-\frac{x^T\bar{C}Ax}{V^2(x)} \le K_1 \quad \text{and} \quad \frac{(x^T\bar{C}\beta)^2 - x^T\bar{C}Ax}{V^2(x)} \le K_1,$$

where K_1 is a positive constant, while

$$\frac{x^T\bar{C}b}{V(x)} \ge \hat{b} \quad \text{and} \quad \frac{(x^T\bar{C}\beta)^2}{V^2(x)} \le \check{\beta}^2.$$

Hence

$$\begin{aligned} & z\big[-y^2x^T\bar{C}(b+Ax) + y^3(x^T\bar{C}\beta)^2\big] - \tfrac{1}{2}(1-\theta)y^4(x^T\bar{C}\beta)^2 \\ & \quad \le K_1(1+y) - [\hat{b} - \tfrac{1}{2}(1+\theta)\check{\beta}^2]y^2. \end{aligned}$$

Substituting this into (3.16) yields

$$\begin{aligned} dz^\theta(t) &\le \theta z^{\theta-2}(t)\Big(K_1(1+y(t)) - [\hat{b} - \tfrac{1}{2}(1+\theta)\check{\beta}^2]y^2(t)\Big)dt \\ &- \theta z^{\theta-1}(t)y^2(t)x^T(t)\bar{C}\beta dw(t). \end{aligned} \tag{3.17}$$

Now, choose $\varepsilon > 0$ sufficiently small for

$$\frac{\varepsilon}{\theta} < [\hat{b} - \tfrac{1}{2}(1+\theta)\check{\beta}^2]. \tag{3.18}$$

Then, by the Itô formula,

$$\begin{aligned} & d[e^{\varepsilon t}z^\theta(t)] = e^{\varepsilon t}[\varepsilon z^\theta(t)dt + dz^\theta(t)] \\ \le\ & \theta\varepsilon^{\varepsilon t}z^{\theta-2}(t)\Big(\frac{\varepsilon}{\theta}(1+y(t))^2 + K_1(1+y(t)) - [\hat{b} - \tfrac{1}{2}(1+\theta)\check{\beta}^2]y^2(t)\Big)dt \\ -\ & \varepsilon^{\varepsilon t}\theta z^{\theta-1}(t)y^2(t)x^T(t)\bar{C}\beta dw(t). \end{aligned}$$

It is easy to see that there is a constant K_2 such that

$$\theta(1+y)^{\theta-2}\Big(\frac{\varepsilon}{\theta}(1+y)^2 + K_1(1+y) - [\hat{b} - \tfrac{1}{2}(1+\theta)\check{\beta}^2]y^2\Big) \le K_2 \tag{3.19}$$

on $y > 0$. Thus

$$d[e^{\varepsilon t} z^\theta(t)] \le K_2 \varepsilon^{\varepsilon t} dt - \varepsilon^{\varepsilon t}\theta z^{\theta-1}(t) y^2(t) x^T(t)\bar{C}\beta dw(t).$$

This implies

$$\mathbb{E}[e^{\varepsilon t} z^\theta(t)] \le z^\theta(0) + \frac{K_2}{\varepsilon}\varepsilon^{\varepsilon t},$$

whence

$$\limsup_{t\to\infty} \mathbb{E}[z^\theta(t)] \le \frac{K_2}{\varepsilon}. \tag{3.20}$$

Moreover, using (3.19), we observe from (3.17) that

$$dz^\theta(t) \le K_2 dt - \theta z^{\theta-1}(t) y^2(t) x^T(t)\bar{C}\beta dw(t).$$

This implies that

$$\begin{aligned} &\mathbb{E}\Big(\sup_{t\le u\le t+1} z^\theta(u)\Big) \\ &\le \mathbb{E}[z^\theta(t)] + K_2 + \mathbb{E}\Big(\sup_{t\le u\le t+1}\Big|\int_t^u \theta z^{\theta-1}(r) y^2(r) x^T(r)\bar{C}\beta dw(r)\Big|\Big). \end{aligned} \tag{3.21}$$

But, by the well-known Burkholder–Davis–Gundy inequality (see e.g. [19, 22]), we compute

$$\begin{aligned} &\mathbb{E}\Big(\sup_{t\le u\le t+1}\Big|\int_t^u \theta z^{\theta-1}(r) y^2(r) x^T(r)\bar{C}\beta dw(r)\Big|\Big) \\ \le\ & 3\theta\mathbb{E}\Big(\int_t^{t+1} z^{2\theta-2}(r) y^4(r)(x^T(r)\bar{C}\beta)^2 dr\Big)^{\frac{1}{2}} \\ \le\ & 3\theta\mathbb{E}\Big(\int_t^{t+1} z^{2\theta}(r)\frac{(x^T(r)\bar{C}\beta)^2}{V^2(x(r))} dr\Big)^{\frac{1}{2}} \\ \le\ & 3\theta\hat{b}\mathbb{E}\Big(\Big[\sup_{t\le r\le t+1} z^\theta(r)\Big]\int_t^{t+1} z^\theta(r) dr\Big)^{\frac{1}{2}} \\ \le\ & \tfrac{1}{2}\mathbb{E}\Big[\sup_{t\le r\le t+1} z^\theta(r)\Big] + \frac{9\theta^2\check{\beta}^2}{2}\mathbb{E}\int_t^{t+1} z^\theta(r) dr. \end{aligned}$$

Substituting this into (3.21) gives

$$\mathbb{E}\Big(\sup_{t\le u\le t+1} z^\theta(u)\Big) \le 2\mathbb{E}[z^\theta(t)] + 2K_2 + 9\theta^2\check{\beta}^2\int_t^{t+1}\mathbb{E}z^\theta(r)dr.$$

Letting $t \to \infty$ and using (3.20) we obtain that

$$\limsup_{t\to\infty} \mathbb{E}\Big(\sup_{t\le u\le t+1} z^\theta(u)\Big) \le K_2(2+(2+9\theta^2\check{\beta}^2)/\varepsilon). \tag{3.22}$$

From this can we show, in the same way as (3.4) was proved, that

$$\limsup_{t\to\infty} \frac{\log(z^\theta(t))}{\log t} \le 1 \quad a.s.$$

This implies

$$\limsup_{t\to\infty} \frac{\log(y(t))}{\log t} \le \frac{1}{\theta} \quad a.s.$$

which further implies, by recalling the definition of $y(t)$,

$$\liminf_{t\to\infty} \frac{\log(V(x(t)))}{\log t} \ge -\frac{1}{\theta} \quad a.s.$$

Since $V(x(t)) \le |C||x(t)|$, we then have

$$\liminf_{t\to\infty} \frac{\log(|x(t)|)}{\log t} \ge -\frac{1}{\theta} \quad a.s.$$

But this holds for any θ that obeys (3.15), we must therefore have the assertion (3.13). □

Theorem 3.3 *Assume that conditions (3.1) and (3.12) hold. Then for any given initial value $x(0) \in \mathbb{R}^n_+$, the solution $x(t)$ of equation (1.3) obeys*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t |x(u)|du \le \frac{2|C|}{\lambda}(\check{b} - \tfrac{1}{2}\hat{\beta}^2) \quad a.s. \tag{3.23}$$

and

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t |x(u)|du \ge \frac{2\hat{c}}{\hat{\lambda}}(\hat{b} - \tfrac{1}{2}\check{\beta}^2) > 0 \quad a.s. \tag{3.24}$$

where $\check{b} = \max_{1\le i\le n} b_i$, $\hat{\beta}^2 = \min_{1\le i\le n} \beta_i^2$, $\hat{c} = \min_{1\le i\le n} c_i$ and

$$\hat{\lambda} = \lambda^+_{\max}(-\bar{C}A - A^T\bar{C}).$$

Proof. Let $V(x)$ be the same as before. It is easy to observe from Theorems 3.1 and 3.2 that

$$\lim_{t\to\infty} \frac{1}{t}\log(V(x(t))) = 0 \quad a.s. \tag{3.25}$$

By the Itô formula, we derive from (3.5) that

$$\begin{aligned} d[\log(V(x(t)))] &= \Big(\frac{x^T(t)\bar{C}b}{V(x(t))} + \frac{x^T(t)\bar{C}Ax(t)}{V(x(t))} - \frac{(x^T(t)\bar{C}\beta)^2}{2V^2(x(t))}\Big)dt \\ &+ \frac{x^T(t)\bar{C}\beta}{V(x(t))}dw(t). \end{aligned} \tag{3.26}$$

By condition (3.1) and the definitions of $\check{b}$ and $\hat{\beta}^2$, we then see

$$d[\log(V(x(t)))] \quad \leq \quad \Big(\check{b} - \frac{\lambda}{2|C|}|x(t)| - \tfrac{1}{2}\hat{\beta}^2\Big)dt + \frac{x^T(t)\bar{C}\beta}{V(x(t))}dw(t). \quad (3.27)$$

Hence

$$\log(V(x(t))) + \frac{\lambda}{2|C|}\int_0^t |x(u)|du$$
$$\leq \quad \log(V(x(0))) + (\check{b} - \tfrac{1}{2}\hat{\beta}^2)t + \int_0^t \frac{x^T(u)\bar{C}\beta}{V(x(u))}dw(u). \quad (3.28)$$

However, it is straightforward to show by the strong law of large numbers of martingales (see e.g. [19, Theorem 3.4 on page 12]) that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{x^T(u)\bar{C}\beta}{V(x(u))}dw(u) = 0 \quad a.s.$$

We can therefore divide both sides of (3.28) by t and then let $t \to \infty$ to obtain

$$\frac{\lambda}{2|C|}\Big(\limsup_{t\to\infty}\frac{1}{t}\int_0^t |x(u)|du\Big) \leq \check{b} - \tfrac{1}{2}\hat{\beta}^2 \quad a.s.$$

which implies the required assertion (3.23). To prove the another assertion, we observe from (3.26) that

$$d[\log(V(x(t)))] \quad \geq \quad \Big(\hat{b} - \frac{\hat{\lambda}|x(t)|^2}{2V(x(t))} - \tfrac{1}{2}\check{\beta}^2\Big)dt + \frac{x^T(t)\bar{C}\beta}{V(x(t))}dw(t). \quad (3.29)$$

Noting that

$$\hat{\lambda} = \lambda^+_{\max}(-\bar{C}A - A^T\bar{C}) \geq -\lambda^+_{\max}(\bar{C}A + A^T\bar{C}) \geq \lambda > 0$$

and $V(x(t)) \geq \hat{c}|x(t)|$, we then have

$$\log(V(x(t))) + \frac{\hat{\lambda}}{2\hat{c}}\int_0^t |x(u)|du$$
$$\geq \quad \log(V(x(0))) + (\hat{b} - \tfrac{1}{2}\check{\beta}^2)t + \int_0^t \frac{x^T(u)\bar{C}\beta}{V(x(u))}dw(u). \quad (3.30)$$

Dividing both sides by t and then letting $t \to \infty$ yields the another assertion (3.24). □

Theorem 3.3 shows that the average in time of the norm of the solution of equation (1.3) is bounded with probability one.

4 Case Studies

In the proof of Theorem 3.3 we have used conditions (3.1) etc. to estimate so that we only have inequality (3.26). However, in some special cases, we may have an equality in order to have more precise result than (3.23).

4.1 One-dimensional case

Let us consider the one-dimensional stochastic population system

$$dx(t) = x(t)[(b - ax(t))dt + \beta dw(t)], \tag{4.1}$$

where b, a and β are all positive numbers. Given the initial value $x(0) > 0$, the solution will remain positive for all $t \geq 0$. Set $y(t) = 1/x(t)$. Then

$$dy(t) = [a + (-b + \beta^2)y(t)]dt - \beta y(t)dw(t).$$

This equation has its explicit solution

$$y(t) = \exp[(-b + \tfrac{1}{2}\beta^2)t - \beta w(t)]\Big(y(0) + a\int_0^t \exp[(b - \tfrac{1}{2}\beta^2)s + \beta w(s)]ds\Big).$$

Hence, equation (4.1) has the explicit solution

$$x(t) = \frac{\exp[(b - \frac{1}{2}\beta^2)t + \beta w(t)]}{x^{-1}(0) + a\int_0^t \exp[(b - \frac{1}{2}\beta^2)s + \beta w(s)]ds}. \tag{4.2}$$

By Theorem 3.1, the solution of equation (4.1) obeys

$$\limsup_{t\to\infty} \frac{\log(x(t))}{\log t} \leq 1 \quad a.s. \tag{4.3}$$

Let us furthermore assume that $b > \frac{1}{2}\beta^2$. Then, by Theorem 3.2, the solution of equation (4.1) also obeys

$$\liminf_{t\to\infty} \frac{\log(x(t))}{\log t} \geq -\frac{\beta^2}{2b - \beta^2} \quad a.s. \tag{4.4}$$

Consequently,

$$\lim_{t\to\infty} \frac{1}{t}\log(x(t)) = 0 \quad a.s. \tag{4.5}$$

By the Itô formula, it is easy to show that

$$\log(x(t)) = \log(x(0)) + (b - \tfrac{1}{2}\beta^2)t - a\int_0^t x(u)du + \beta w(t).$$

Dividing both sides by t and then letting $t \to \infty$ we obtain

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x(u)du = \frac{b - \frac{1}{2}\beta^2}{a} \quad a.s. \tag{4.6}$$

4.2 Multi-dimensional system of facultative mutualism

Let us now return to equation (1.3) but impose the following condition

$$a_{ii} < 0, \quad a_{ij} \geq 0, \quad 1 \leq i, j \leq n, \ i \neq j. \tag{4.7}$$

In such a system, each species enhances the growth of the other, as the parameters $a_{ij} \geq 0$ $(i \neq j)$. This type of ecological interaction is known as facultative mutualism (see e.g. [11, 21]). We assume that

$$-A \text{ is a non-singular M-matrix.} \tag{4.8}$$

Non-singular M-matrices have many very nice properties (see e.g. [5, 22]). In particular, (4.7) is equivalent to one of the followings:

(i) There is a positive-definite diagonal matrix $\bar{C} = \text{diag}(c_1, \cdots, c_n)$ such that $-(\bar{C}A + A^T\bar{C})$ is a positive-definite matrix.

(ii) For any $y \in \mathbb{R}^n_+$, the linear equation $y + Ax = 0$ has a unique solution $x \in \mathbb{R}^n_+$.

Property (i) shows that

$$\lambda^+_{\max}(\bar{C}A + A^T\bar{C}) \leq \lambda_{\max}(\bar{C}A + A^T\bar{C}) < 0.$$

By Theorem 3.1, we observe that, under condition (4.8), the solution of equation (1.3) obeys

$$\limsup_{t\to\infty} \frac{\log(|x(t)|)}{\log t} \leq 1 \quad a.s. \tag{4.9}$$

Moreover, we note that, for each $1 \leq i \leq n$,

$$dx_i(t) = x_i(t)\Big[\Big(b_i + \sum_{j=1}^{n} a_{ij}x_i(t)\Big)dt + \beta_i dw(t)\Big].$$

Introduce the corresponding stochastic differential equation

$$d\xi_i(t) = \xi_i(t)[(b_i + a_{ii}\xi_j(t))dt + \beta_i dw(t)]$$

with initial value $\xi_i(0) = x_i(0)$. By the classical comparison theorem (see e.g. [12]) we have $x_i(t) \geq \xi_i(t)$ a.s. for all $t \geq 0$. If we further assume that

$$b_i > \tfrac{1}{2}\beta_i^2, \quad 1 \leq i \leq n, \tag{4.10}$$

then, by result (4.4), we have

$$\liminf_{t\to\infty} \frac{\log(x_i(t))}{\log t} \geq -\frac{\beta_i^2}{2b_i - \beta_i^2} \quad a.s. \tag{4.11}$$

Combining the results above we can conclude that, under conditions (4.8) and (4.10), the solution of equation (1.3) obeys

$$\lim_{t\to\infty} \frac{1}{t} \log(x_i(t)) = 0 \quad a.s., \quad 1 \le i \le n. \tag{4.12}$$

Define $\eta_i(t) = \log(x_i(t))$ $(1 \le i \le n)$ and $\eta(t) = (\eta_1(t), \cdots, \eta_n(t))^T$. It is easy to show that

$$d\eta(t) = (\zeta + Ax(t))dt + \beta dw(t), \tag{4.13}$$

where $\zeta = (\zeta_1, \cdots, \zeta_n)^T$ with $\zeta_i = b_i - \frac{1}{2}\beta_i^2$. By condition (4.10), $\zeta \in \mathbb{R}_+^n$, whence by Property (ii), the linear equation

$$\zeta + A\sigma = 0 \tag{4.14}$$

has a unique solution $\sigma \in \mathbb{R}_+^n$. However, it follows from (4.13) that

$$\frac{1}{t}(\eta(t) - \eta(0)) = \zeta + A\Big(\frac{1}{t}\int_0^t x(u)du\Big) + \frac{\beta w(t)}{t}.$$

Noting that $\lim_{t\to\infty} w(t)/t \to 0$ a.s. and using (4.12) we may let $t \to \infty$ to obtain that

$$\zeta + A\Big(\lim_{t\to\infty} \frac{1}{t}\int_0^t x(u)du\Big) = 0 \quad a.s.$$

This, together with (4.14), yields

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x(u)du = \sigma \quad a.s. \tag{4.15}$$

We form the above result as a theorem to conclude our paper.

Theorem 4.1 *Under conditions (4.8) and (4.10), the solution of equation (1.3) obeys (4.15), where $\sigma \in \mathbb{R}_+^n$ is the unique solution to equation (4.14).*

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