STABILISATION AND DESTABILISATION OF NONLINEAR DIFFERENTIAL EQUATIONS BY NOISE

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Abstract. This paper considers the stabilisation and destabilisation by a Brownian noise perturbation which preserves the equilibrium of the ordinary differential equation $x'(t) = f(x(t))$. In an extension of earlier work, we lift the restriction that f obeys a global linear bound, and show that when f is locally Lipschitz, a function g can always be found so that the noise perturbation $g(X(t))\, dB(t)$ either stabilises an unstable equilibrium, or destabilises a stable equilibrium. When the equilibrium of the deterministic equation is non–hyperbolic, we show that a non–hyperbolic perturbation suffices to change the stability properties of the solution.

1. Introduction

It is well known that noise can be used to stabilise a given unstable system or to make a system even more stable when it is already stable. Stabilisation by deterministic periodic "noise" has been studied a great deal by many authors, e.g. Bellman et al. [7], Kushner [14], Meerkov [21], and Zhabko and Kharitonov [27]. Deterministic vibrational stabilisation for delay systems was investigated in Lehman *et al.* [15]. In the case of random noise, the pioneering work is due to Hasminskii [11, p.229], who stabilised a system by using two white noise sources. Later, Arnold et al. [6] showed, in particular, that the system $\dot{x}(t) = Ax(t)$ can be stabilised by zero mean stationary parameter noise if and only if trace $A < 0$. On the other hand, in the nonlinear case, Scheutzow [24] provided some examples on stabilisation and destabilisation in the plane, and Mao [18] developed a general theory on stabilisation and destabilisation by Brownian motion. Recently, Appleby [1, 2], Appleby and Mao [4], and Appleby and Flynn [3] have extended these results on stabilisation and destabilisation to functional differential equations and Volterra equations. The stabilisation of general scalar nonlinear stochastic difference equations has been examined in Appleby, Mao and Rodkina [5]. General rates of decay in

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stabilised nonlinear differential equations have been considered in a paper of Caraballo, Garrido-Atienza and Real [8]. We should also mentioned that some results on stabilisation by random switching are given in Verriest [25, 26] while stochastic stabilisation for partial differential equations are given in Caraballo *et al.* [9, 10].

The present paper may be thought of as an extension of the work of [18] to general nonlinear equations, and draws on some of the techniques developed for nonlinear stochastic difference equations in [5]. In [18], Mao showed that a general finite dimensional ordinary differential equation

$$\dot{x}(t) = f(x(t)), \quad t > 0; \quad x(0) = \xi \in \mathbb{R}^d \tag{1.1}$$

could be stabilised (resp., destabilised) using a Brownian motion. More precisely, he showed that the equilibrium solution of the SDE

$$dX(t) = f(X(t))\,dt + g(X(t))\,dB(t) \tag{1.2}$$

is almost surely asymptotically stable (resp., unstable), for an appropriate choice of g. (For the general theory of SDEs we refer the reader to [19, 22, 23].) In that paper, he assumes that f and g obey a global linear bound of the form

$$|f(x)| \vee |g(x)| \leq K_0|x|, \quad x \in \mathbb{R}^d, \tag{1.3}$$

for some $K_0 > 0$. Indeed, under some further conditions, (1.3) can be weakened to a one–sided growth condition of the form

$$\langle x, f(x)\rangle + \frac{1}{2}|g(x)|^2 \leq K_1|x|^2, \quad x \in \mathbb{R}^d \tag{1.4}$$

(see e.g. Mao *et al.* [20]). Although the condition (1.4) allows for a wider class of SDEs to be studied than (1.3), and in particular, means that we can study equations where neither f nor g obey a global linear bound, many deterministic dynamical systems with equilibria at 0 are still excluded. For example, consider the system (1.1) with

$$f(x) = \log(1 + x)x. \tag{1.5}$$

Clearly, a condition of the form (1.4) cannot hold, even though solutions of (1.1) are well defined on $(0, \infty)$. In this work, we show that the analysis can be extended to cover cases of this type.

It is also assumed in [18] that the stabilising noise is (essentially) of the form $\Sigma X(t)\,dB(t)$, or that the diffusion term in (1.2) does not depart too much from linearity. Certainly, if the deterministic system (1.1) at 0 is sublinear, such a perturbation suffices to stabilise the unstable deterministic equation. However, one may not need such a strong noise perturbation in order to stabilise the solution. Therefore, we ask here what the critical size of stochastic perturbation $g(X(t))\,dB(t)$ at zero must be in order to stabilise the deterministic equation (1.1). On the other hand, if a system is asymptotically stable, but the equilibrium is non-hyperbolic, we may be able to destabilise using a relatively weak noise perturbation.

A final feature distinguishes this analysis from that of [18]. There, because f obeys either a global linear bound or a one–sided growth condition, solutions of (1.1) cannot explode in finite time. However, if such a restriction is lifted, solutions of (1.1) can explode in finite time. As a concrete example, consider equation (1.1) where f obeys (1.5) and once again $f : [0, \infty) \to [0, \infty)$. If f obeys

$$\int_1^\infty \frac{du}{uf(u^{1/2})} < \infty,$$

then the solution explodes in finite time; in fact

$$\lim_{t \uparrow T} |x(t)| = \infty, \quad \text{where } T = \int_{|x(0)|^2}^\infty \frac{du}{uf(u^{1/2})}.$$

In this paper, we show that stabilisation is still possible with such functions f in (1.1).

The main results of the paper are the following. First, given an equation (1.1) where f is locally Lipschitz continuous, it is always possible to design a noise perturbation g such that all solutions of (1.2) tend to zero, almost surely. Necessary and sufficient conditions on g (in terms of f) for which this is true are known in the scalar case. However, the method presented here can be applied to a wider class of stochastic dynamical systems (such as non–autonomous SDEs and even stochastic functional differential equations). Moreover, the conditions given recover both the earlier results in [18] and sharp sufficient conditions supplied by the Feller explosion theory in $\mathbb{R}$ (see e.g. [12, p.348]).

We are also able to show that given an equation (1.1) with a (globally or locally) asymptotically stable solution and where f is locally Lipschitz continuous, it is always possible to design a noise perturbation g such that no solutions of (1.2) tends to zero, almost surely. Under some other conditions, we can even show that solutions must tend to infinity, a.s.

The structure of the paper is as follows: the next section introduces notation, results required from the literature, and a precise statement of the problem to be studied, and standing assumptions required. Section 3 deals with stabilisation by noise, and Section 4 with destabilisation by noise when the system is of dimension two or higher. We show that finite dimensional systems can always be stabilised, and that all systems of dimension two or higher can be destabilised. However, we cannot destabilise a globally asymptotically stable scalar system with equilibrium preserving noise, and this result is the subject of Section 5. The final section (Section 6) shows that the solution of the stochastic differential equation cannot hit zero, the proof having been postponed in Section 2.

2. Preliminaries

2.1. **Notation and required results.** Throughout the paper, unless otherwise specified, we will employ the following notation. Let $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}(0)$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), \cdots, B_r(t))^*$, $t \geq 0$, be an r-dimensional Brownian motion defined on the probability space, where $*$ denotes the transpose of a vector or matrix. If x, y are real numbers, then $x \vee y$ denotes the maximum of x and y, and $x \wedge y$ denotes the minimum of x and y. Let $|x|$ be the Euclidean norm of a vector $x \in \mathbb{R}^d$ and $\langle x, y\rangle$ be the inner product of vectors $x, y \in \mathbb{R}^d$, while, without any confusion, the notation $\langle\cdot\rangle$ is used for the quadratic variation of a martingale. Vectors $x \in \mathbb{R}^d$ are thought as column ones so to get row vectors we use x^*. Let $\mathbf{e}_i \in \mathbb{R}^d$ denote the i-th standard basis vector in $\mathbb{R}^d$. The space of $d\times r$ matrices with real entries is denoted by $\mathbb{R}^{d\times r}$. If $A = (a_{ij})$ is a $d \times r$ matrix, we denote its Frobenius norm by

$$|A|_F^2 = \sum_{i=1}^{d}\sum_{j=1}^{r} a_{ij}^2.$$

If A is a $d \times d$ matrix with entries $a_{ij} = \alpha_i$ for $i = j$ and $a_{ij} = 0$ otherwise, we write $A = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_d)$.

The following result (see. e.g. [16, Theorem 7 on p.139]) will be of great use in the paper. We introduce the notation $\{Z \rightarrow\}$ to denote the set of all $\omega \in \Omega$ for which the scalar process Z has the property that $\lim_{t\to\infty} Z(t)$ exists and is finite. We denote the limit in this case by $Z(\infty)$.

Lemma 2.1. *Let A_1 and A_2 be a.s. non-decreasing processes. Let also Z be a non–negative semimartingale, with $\mathbb{E}(Z) < \infty$ and*

$$Z(t) = Z(0) + A_1(t) - A_2(t) + M(t), \quad t \geq 0$$

where M is a local martingale. Then

$$\{\omega : A_1(\infty) < \infty\} \subseteq \{Z \rightarrow\} \cap \{\omega : A_2(\infty) < \infty\} \qquad a.s.$$

2.2. **Statement of the problem.** We study the asymptotic behaviour of solutions of the SDE (1.2). (For the advanced background on SDEs we refer the reader to [19, 22, 23].) We denote the solution of this initial value problem by $X(\cdot, \xi)$, or simply X. We now impose some hypotheses on f and g that stand throughout the paper.

Assumption 2.2. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}^{d\times r}$ obey that*

$$f(0) = 0, \quad g(0) = 0. \tag{2.1}$$

This ensures that when $\xi = 0$, $X(t,\xi) = 0$ for all $t \geq 0$ is a solution of the SDE (1.2). Therefore, the noise perturbation preserves the equilibrium of the system.

We assume that f and g are locally Lipschitz continuous.

Assumption 2.3. *For every integer $n \geq 1$, there is $K_n > 0$ such that*

$$|f(x) - f(y)| \leq K_n|x - y|, \quad |g(x) - g(y)|_F \leq K_n|x - y| \tag{2.2}$$

for all $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \leq n$.

By virtue of Assumption 2.3, we are guaranteed that there is a unique continuous adapted process X (see e.g. [19, 22, 23]) such that

$$X(t\wedge\tau_k) = \xi + \int_0^{t\wedge\tau_k} f(X(s))\,ds + \int_0^{t\wedge\tau_k} g(X(s))\,dB(s), \quad t \geq 0, \quad \text{a.s.} \tag{2.3}$$

where $\tau_k = \inf\{t > 0 : |X(t)| \geq k\}$. The equation (1.2) has a global solution if the explosion time τ_e^ξ defined by

$$\tau_e^\xi = \lim_{k\to\infty} \tau_k = \inf\{t > 0 : |X(t,\xi)| \notin [0,\infty)\}. \tag{2.4}$$

obeys $\tau_e^\xi = \infty$, a.s.

Throughout the paper, it is important to show that solutions cannot reach zero in finite time. To make our argument more concise, define

$$\vartheta_0^\xi = \inf\{t > 0 : |X(t,\xi)| = 0\}. \tag{2.5}$$

Proposition 2.4. *Suppose that f and g obey Assumptions 2.2 and 2.3. Let X be the unique continuous adapted process that obeys (2.3). With τ_e^ξ and ϑ_0^ξ defined as in (2.4), (2.5), we have $\tau_e^\xi \leq \vartheta_0^\xi$ a.s.*

The proof of this result is relegated to the Appendix.

3. Stabilisation of (1.1)

In this section, we wish to give conditions on g which ensure that solutions of (1.2) will tend to zero almost surely. It turns out that assuming the following about f and g will suffice:

Assumption 3.1. *There exists $\theta \in (0,1)$ such that*

$$|x|^2(2\langle x, f(x)\rangle + |g(x)|_F^2) - (2-\theta)|x^* g(x)|^2 \leq 0, \quad x \in \mathbb{R}^d. \tag{3.1}$$

Assumption 3.2. *For every $L > 0$,*

$$\underline{g}(L) := \min_{|x|=L} |x^* g(x)| > 0. \tag{3.2}$$

Before we can discuss and prove asymptotic stability, it is first necessary to ensure that the solutions are well defined on $[0,\infty)$, a.s. Therefore, we need to prove first that $\tau_e^\xi = \infty$, where τ_e^ξ is defined by (2.4). Assumption 3.1 guarantees that there is a unique global strong solution of (1.2), and this Assumption turns out to be crucial in giving the asymptotic stability of solutions.

Proposition 3.3. *Suppose that f and g obey Assumptions 2.2, 2.3, and 3.1. Then there exists a unique continuous adapted process X which is a solution of (1.2) such that $\tau_e^\xi = \vartheta_0^\xi = \infty$, a.s.*

The proof of this result is given in the Appendix. The statement says that the solution does not explode, but also may not end up at the equilibrium (0).

If Assumption 3.2 also holds, almost sure asymptotic stability is guaranteed.

Theorem 3.4. *Suppose that f and g obey Assumptions 2.2, 2.3, 3.1 and 3.2. Then there exists a unique continuous adapted process X which is a global solution of (1.2) and which obeys*

$$\lim_{t\to\infty} X(t,\xi) = 0, \quad a.s.$$

The proof of this result is also deferred until later in the section.

We notice that this result also shows that when 0 is a globally asymptotically stable equilibrium of (1.1) when $f(0) = 0$, we can preserve this global stability in (1.2). A sufficient condition for global asymptotic stability of the zero solution of (1.2) is

$$\langle x, f(x)\rangle \le -F(|x|) \le 0, \quad x \in \mathbb{R}^d,$$

where $F : [0,\infty) \to [0,\infty)$ is continuous with $F(x) = 0$ only when $x = 0$. Therefore, once g is so small that there exists $\theta' \in (0, 1/2)$ such that

$$F(|x|) \ge \frac{1}{2}|g(x)|_F^2 - (1-\theta')\frac{|x^* g(x)|^2}{|x|^2}, \quad x \in \mathbb{R}^d \setminus \{0\},$$

we have that (3.1) is satisfied (with $\theta = 2\theta'$), and so, if g also obeys (3.2), we have that the solution of (1.2) obeys $\lim_{t\to\infty} X(t,\xi) = 0$ a.s. for any $\xi \in \mathbb{R}^d$.

Before giving examples of functions g which will satisfy these hypotheses, and proving the results stated above, we comment briefly on the assumptions required in this section. In particular, we see that both assumptions are analogues of conditions required in the scalar case, and for this reason are difficult to relax significantly. When $d = 1$ and $r = 1$, Assumption 3.2 reduces to the condition that $g(x) \neq 0$ for $x \neq 0$, which is the standard non–degeneracy assumption in the scalar case, which also ensures that the equilibrium is unique. Also, when $d = 1$ and $r = 1$, Assumption 3.1 reduces to the condition

$$\sup_{|x|>0} \frac{xf(x)}{g^2(x)} < \frac{1}{2}.$$

However, this condition is not far from being necessary to ensure the asymptotic stability of scalar autonomous stochastic differential equations. Indeed, if there exists $L \in [-\infty, \infty]$ such that

$$L := \lim_{x \to 0} \frac{xf(x)}{g^2(x)},$$

it is known (see e.g. [11, p.247]) that $-\infty \le L < 1/2$ ensures that solutions tend to zero with (at least) positive probability, while $L > 1/2$ implies that

$$\mathbb{P}[\{\omega : \lim_{t\to\infty} X(t,\omega) = 0\}] = 0.$$

3.1. **Examples.** We now show that the conditions of Theorem 3.4 are satisfied for many functions g, and, in particular, that for every function f (and therefore for every equation (1.1)), a function g can be chosen to stabilise solutions of (1.1). This example is modelled on an example in [18].

Example 3.5. Let $\gamma : \mathbb{R}^d \to \mathbb{R}$ be locally Lipschitz continuous with $\gamma(x) \neq 0$ for $x \neq 0$. Suppose also that there exists $c > 2$ such that

$$\gamma^2(x) > c\frac{\langle x, f(x)\rangle}{|x|^2}, \quad x \in \mathbb{R}^d \setminus \{0\}.$$

Now let $g : \mathbb{R}^d \to \mathbb{R}^{d\times r}$ be given by

$$g(x) = \gamma(x) x v^*(x), \quad x \in \mathbb{R}^d,$$

where $v : \mathbb{R}^d \to \mathbb{R}^r$ is a locally Lipschitz continuous function such that $|v(x)| = 1$ for all $x \in \mathbb{R}^d$. It is easy to see that the positive definiteness of γ^2 ensures Assumption 3.2 is satisfied. Also, defining $\theta \in (0,1)$ by $1-\theta = 2/c$, we have Assumption 3.1.

By virtue of Example 3.5, the following corollary of Theorem 3.4 holds.

Corollary 3.6. *Suppose that $f : \mathbb{R}^d \to \mathbb{R}^d$ is a locally Lipschitz continuous function with $f(0) = 0$, and that the zero solution of*

$$\dot{x}(t) = f(x(t)), \quad t > 0,$$

is unstable. Then this system can be stabilised by an equilibrium–preserving noise perturbation, in the sense that there exists a locally Lipschitz continuous function $g : \mathbb{R}^d \to \mathbb{R}^{d\times r}$ with $g(0) = 0$ such that, for every $\xi \in \mathbb{R}^d$, the unique continuous adapted process X which obeys

$$X(t) = \xi + \int_0^t f(X(s))\,ds + \int_0^t g(X(s))\,dB(s), \quad t \ge 0, \quad a.s.$$

obeys $\lim_{t\to\infty} X(t) = 0$, *a.s.*

3.2. **Proof of Theorem 3.4.** By Propositions 2.4 and 3.3, we know that the process $|X|^2 = \{|X(t)|^2; t \geq 0\}$ is well-defined and positive. Itô's formula gives

$$|X(t)|^2 = |\xi|^2 + \int_0^t \left(2\langle X(s), f(X(s))\rangle + |g(X(s))|_F^2\right)\, ds + \int_0^t 2\langle X(s), g(X(s))dB(s)\rangle. \tag{3.3}$$

Let $\theta \in (0,1)$ be the number in (3.1). Then, we may apply Itô's rule to the positive process $|X|^2$, using the fact that $|x|^\theta = (|x|^2)^{\theta/2}$ to get

$$|X(t)|^\theta = |\xi|^\theta + \int_0^t \frac{\theta}{2}|X(s)|^{\theta-4}q(X(s))\, ds + \int_0^t \theta|X(s)|^{\theta-2}\langle X(s), g(X(s))dB(s)\rangle,$$

where

$$q(x) = |x|^2(2\langle x, f(x)\rangle + |g(x)|_F^2) + (\theta-2)|x^*g(x)|^2. \tag{3.4}$$

Then, by (3.1), $q(x) \leq 0$ for all $x \in \mathbb{R}^d \setminus \{0\}$. Define $A_1(t) = 0$, and

$$A_2(t) = -\int_0^t \frac{\theta}{2}|X(s)|^{\theta-4}q(X(s))\, ds.$$

Then A_2 is a non-decreasing process. Letting M be the local martingale

$$M(t) = \int_0^t \theta|X(s)|^{\theta-2}\langle X(s), g(X(s))dB(s)\rangle, \quad t \geq 0,$$

whose quadratic variation is given by

$$\langle M\rangle(t) = \int_0^\infty \theta^2|X(s)|^{2\theta-4}|X^*(s)g(X(s))|^2\, ds,$$

we see that

$$|X(t)|^\theta = |\xi|^\theta + A_1(t) - A_2(t) + M(t), \quad t \geq 0.$$

Applying Lemma 2.1 to the positive semimartingale $Z = |X|^\theta$ we can conclude that

$$\lim_{t\to\infty}|X(t)|^\theta \quad \text{and} \quad \lim_{t\to\infty}A_2(t) \quad \text{exist and are finite, a.s.}$$

Therefore $\lim_{t\to\infty} M(t)$ exists and is finite, a.s. Hence $\lim_{t\to\infty}\langle M\rangle(t)$ exists and is finite, a.s., which reads as:

$$\int_0^\infty \theta^2|X(s)|^{2\theta-4}|X^*(s)g(X(s))|^2\, ds < \infty, \quad \text{a.s.}$$

Let $\Omega_1 = \{\omega \in \Omega : \lim_{t\to\infty}|X(t,\omega)| = \hat{x}(\omega) > 0\}$, and suppose that $\mathbb{P}[\Omega_1] > 0$. Then on Ω_1 we must have

$$\int_0^\infty |X^*(s)g(X(s))|^2\, ds < \infty, \quad \text{a.s.}$$

By Assumption 3.2, we have for each $\omega \in \Omega_1$ that

$$\liminf_{t\to\infty} |X^*(s)g(X(s))|^2(\omega) \geq \underline{g}^2(\hat{x}(\omega)) > 0.$$

Therefore, for each $\omega \in \Omega_1$, we have that

$$\left(\int_0^\infty |X^*(s)g(X(s))|^2\,ds\right)(\omega) = \infty,$$

which contradicts a consequence of the supposition that $\mathbb{P}[\Omega_1] > 0$. Hence $\mathbb{P}[\Omega_1] = 0$. If we define $\Omega_2 = \{\omega \in \Omega : \lim_{t\to\infty} |X(t,\omega)| = 0\}$, we already know that $\mathbb{P}[\Omega_1 \cup \Omega_2] = 1$. Hence $\mathbb{P}[\Omega_2] = 1$, as required.

4. Destabilisation of (1.1)

In this section, we wish to give conditions on g which ensure that solutions of (1.2) cannot tend to zero with positive probability. It transpires that assuming the following about f and g suffices.

Assumption 4.1. *There exists $\theta \in (0,1)$ such that*

$$|x|^2(2\langle x, f(x)\rangle + |g(x)|_F^2) - (2+\theta)|x^*g(x)|^2 \geq 0, \quad x \in \mathbb{R}^d \setminus \{0\}. \tag{4.1}$$

Define $\Omega_3 = \{\omega \in \Omega : \tau_e^\xi(\omega) < \infty\}$. Then, on Ω_3, we have that

$$\lim_{t\uparrow\tau_e^\xi} |X(t,\xi)| = \infty,$$

and X is not defined on $[\tau_e^\xi, \infty)$. Clearly

$$\Omega_4 := \{\omega \in \Omega : \liminf_{t\to\tau_e^\xi} |X(t)| > 0\} \supseteq \Omega_3.$$

We now want to show whenever $\omega \in \overline{\Omega}_3 (:= \Omega \setminus \Omega_3)$, we have $\omega \in \Omega_4$. This implies that $\overline{\Omega}_3 \subseteq \Omega_4$, and hence that $\mathbb{P}[\Omega_4] = 1$.

If Assumption 4.1 now holds, solutions cannot be asymptotically stable with positive probability.

Theorem 4.2. *Let $\xi \in \mathbb{R}^d \setminus \{0\}$ and suppose that f and g obey Assumptions 2.2, 2.3, and 4.1, and let τ_e^ξ be defined by (2.4). Let X be the unique continuous adapted process which is a solution of (1.2) on $[0,\tau_e^\xi)$. Then X obeys*

$$\liminf_{t\uparrow\tau_e^\xi} |X(t,\xi)| > 0, \quad \textit{a.s.}$$

The proof of this result is deferred until later in the section.

This result shows that when 0 is an unstable equilibrium of (1.1) (where $f(0) = 0$), we can preserve this instability in (1.2). A sufficient condition for the instability of the zero solution of (1.2) is

$$\langle x, f(x)\rangle \geq F(|x|) \geq 0, \quad x \in \mathbb{R}^d, \tag{4.2}$$

where $F : [0,\infty) \to [0,\infty)$ is continuous with $F(x) = 0$ only when $x = 0$. Therefore, once g is so small that there exists $\theta' > 0$ such that

$$F(|x|) \geq (1+\theta')\frac{|x^* g(x)|^2}{|x|^2} - \frac{1}{2}|g(x)|_F^2 \geq 0, \quad x \in \mathbb{R}^d \setminus \{0\},$$

then (4.1) is satisfied (with $\theta = 2\theta'$), and so the solution of (1.2) obeys $\liminf_{t\uparrow\tau_e^\xi} |X(t,\xi)| > 0$, a.s. for any $\xi \in \mathbb{R}^d \setminus \{0\}$, by Theorem 4.2.

Theorem 4.2 also shows that when 0 is a stable equilibrium of (1.1) (where $f(0) = 0$), we can destabilise it by noise. Let us illustrate this by examples.

4.1. **Examples.** We first study an example with dimension $d \geq 2$, which is modelled on an example in [18].

Example 4.3. Let $d \geq 2$ and choose $r = d$. Let $\gamma : \mathbb{R}^d \to \mathbb{R}$ be locally Lipschitz continuous with $\gamma(x) \neq 0$ for $x \neq 0$. Suppose also that there is $c > 6$ such that

$$\gamma^2(x) + c\frac{\langle x, f(x)\rangle}{|x|^2} \geq 0, \quad x \in \mathbb{R}^d \setminus \{0\}.$$

Now, using the convention that $x = (x_1, x_2, \dots, x_d)$, and $x_{d+1} = x_1$, define $g : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ by $g(x) = \gamma(x)\mathrm{diag}(x_2, x_3, ..., x_d, x_{d+1})$, $x \in \mathbb{R}^d$. Then,

$$|x^* g(x)|^2 = \gamma^2(x)\sum_{i=1}^d x_i^2 x_{i+1}^2$$

and$|g(x)|_F^2 = \gamma^2(x)|x|^2$ for all $x \in \mathbb{R}^d$. So for $x \in \mathbb{R}^d \setminus \{0\}$

$$\begin{aligned} &|x|^2(2\langle x, f(x)\rangle + |g(x)|_F^2) - (2+\theta)|x^* g_i(x)|^2 \\ &\qquad = 2|x|^2\langle x, f(x)\rangle + \gamma^2(x)\left\{|x|^4 - (2+\theta)\sum_{i=1}^d x_i^2 x_{i+1}^2\right\}. \end{aligned}$$

Now, proceeding as in [18], we obtain the following sequence of inequalities

$$\sum_{i=1}^d x_i^2 x_{i+1}^2 \leq \sum_{i=1}^d \frac{1}{2}(x_i^4 + x_{i+1}^4) = \sum_{i=1}^d x_i^4.$$

Thus

$$3\sum_{i=1}^d x_i^2 x_{i+1}^2 \leq 2\sum_{i=1}^d x_i^2 x_{i+1}^2 + \sum_{i=1}^d x_i^4 \leq \left(\sum_{i=1}^d x_i^2\right)^2 = |x|^4.$$

Hence

$$|x|^4 - (2+\theta)\sum_{i=1}^d x_i^2 x_{i+1}^2 \geq \left(1 - \frac{2+\theta}{3}\right)|x|^4 = \frac{1-\theta}{3}|x|^4,$$

and so

$$|x|^2(2\langle x, f(x)\rangle + |g(x)|_F^2) - (2+\theta)|x^* g_i(x)|^2 \\ \geq |x|^4 \left(\frac{2\langle x, f(x)\rangle}{|x|^2} + \gamma^2(x)\frac{1-\theta}{3} \right) \geq 0,$$

where we have chosen $\theta \in (0,1)$ so that $1-\theta = 6/c$. Hence Assumption 4.1 is satisfied and Theorem 4.2 applies. Therefore, we can see that it is possible to destabilise a globally stable solution (in the case when $\langle x, f(x)\rangle < 0$ for all $x \in \mathbb{R}^d$).

Next, we consider an even-dimensional example, which once again is inspired by an example in [18]. This example provides an alternative, which allows r to be different from d, for the even-dimensional case only.

Example 4.4. Let $m \in \mathbb{N}$ and $d = 2m$. Let $\gamma : \mathbb{R}^d \to \mathbb{R}$ be locally Lipschitz continuous with $\gamma(x) \neq 0$ for $x \neq 0$. Suppose also that

$$\gamma^2(x) + 2\frac{\langle x, f(x)\rangle}{|x|^2} \geq 0, \quad x \in \mathbb{R}^d \setminus \{0\}.$$

To design the destabilising perturbation $g : \mathbb{R}^d \to \mathbb{R}^{d\times r}$, let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and introduce the $d \times d$ matrix Σ by

$$\Sigma = \mathrm{diag}(J, J, \dots, J).$$

Now, let

$$g(x) = \gamma(x)\Sigma x v^*(x), \quad x \in \mathbb{R}^d,$$

where $v : \mathbb{R}^d \to \mathbb{R}^r$ is a locally Lipschitz continuous function such that $|v(x)| = 1$ for all $x \in \mathbb{R}^d$. Hence $|x^* g(x)| = 0$, $|g(x)|_F^2 = \gamma(x)^2|x|^2$ for all $x \in \mathbb{R}^d$, and so for $x \in \mathbb{R}^d \setminus \{0\}$

$$|x|^2(2\langle x, f(x)\rangle + |g(x)|_F^2) - (2+\theta)|x^* g(x)|^2 \\ = |x|^2(2\langle x, f(x)\rangle + \gamma^2(x)|x|^2) \geq 0.$$

Consequently, Assumption 4.1 is satisfied and Theorem 4.2 applies. Therefore, we can see that it is possible to destabilise a globally stable solution (in the case when $\langle x, f(x)\rangle < 0$ for all $x \in \mathbb{R}^d$) when the dynamical system is even–dimensional.

Taking the results of Examples 4.3 and 4.4 together, we have the following Corollary of Theorems 4.2.

Corollary 4.5. *Let $d \geq 2$. Suppose that $f : \mathbb{R}^d \to \mathbb{R}^d$ is a locally Lipschitz continuous function with $f(0) = 0$, and that the zero solution of*

$$\dot{x}(t) = f(x(t)), \quad t > 0$$

is asymptotically stable. Then this system can be destabilised by an equilibrium–preserving noise perturbation, in the sense that there exists a locally Lipschitz continuous function $g : \mathbb{R}^d \to \mathbb{R}^{d\times r}$ *with* $g(0) = 0$ *(we may have to choose* $r = d$ *when* d *is odd) such that, for every* $\xi \in \mathbb{R}^d$*, the unique continuous adapted process* X *which obeys*

$$X(t) = \xi + \int_0^t f(X(s))\,ds + \int_0^t g(X(s))\,dB(s), \quad t \geq 0, \quad a.s.$$

obeys $\liminf_{t\uparrow\tau_e^\xi} X(t) > 0$*, a.s., where* $\tau_e^\xi = \inf\{t > 0 : |X(t,\xi)| \notin [0,\infty)\}$*.*

4.2. **Proof of Theorem 4.2.** We see that it is necessary to prove only that $\overline{\Omega}_3 \subseteq \Omega_4$. Thus from here on, we work only on $\overline{\Omega}_3$. We may assume that $\overline{\Omega}_3$ has positive probability, for if it does not, then $\mathbb{P}[\Omega_3] = 1$, which gives $\mathbb{P}[\Omega_4] = 1$ automatically, and the result is proved.

On $\overline{\Omega}_3$, we have (3.3). Therefore, by Proposition 2.4, using Itô's rule, we get

$$\log|X(t)|^2 = \log|\xi|^2 + \int_0^t |X(s)|^{-4} r(X(s))\,ds + \int_0^t \frac{2\langle X(s), g(X(s))dB(s)\rangle}{|X(s)|^2}$$

where $r(x) = |x|^2(2\langle x, f(x)\rangle + |g(x)|^2) - 2|x^* g(x)|^2 \geq \theta|x^* g(x)|^2$, by (4.1). Hence

$$\log|X(t)|^2 \geq \log|\xi|^2 + \theta\int_0^t \frac{|X^*(s)g(X(s))|^2}{|X(s)|^4}\,ds + \int_0^t \frac{2\langle X(s), g(X(s))dB(s)\rangle}{|X(s)|^2},$$

or, with the local martingale M defined by

$$M(t) = \int_0^t \frac{2\langle X(s), g(X(s))dB(s)\rangle}{|X(s)|^2},$$

we have

$$\log|X(t)|^2 \geq \log|\xi|^2 + \frac{\theta}{4}\langle M\rangle(t) + M(t). \tag{4.3}$$

There are two cases to consider. Define

$$\Omega_5 = \{\omega \in \overline{\Omega}_3 : \lim_{t\to\infty}\langle M\rangle(t,\omega) < \infty\}$$

and so $\overline{\Omega}_5 = \{\omega \in \overline{\Omega}_3 : \lim_{t\to\infty}\langle M\rangle(t,\omega) = \infty\}$.

By the definition of the quadratic variation of a continuous local martingale (see e.g. [16, 12]),

$$\bar{M}(t) := M^2(t) - \langle M\rangle(t)$$

is a continuous local martingale. Writing this as

$$M^2(t) = \bar{M}(t) + \langle M\rangle(t)$$

and then applying Lemma 2.1, we see that for almost all $\omega \in \Omega_5$, $\lim_{t\to\infty} M^2(t,\omega)$, whence $\lim_{t\to\infty} M(t,\omega)$ exists and is finite almost surely, since $M(t)$ is continuous. Hence, by (4.3) $\liminf_{t\to\infty} \log|X(t)|^2$ is bounded below by a finite random variable a.s. on Ω_5, and so $\liminf_{t\to\infty} |X(t)| > 0$, a.s. on Ω_5.

On the other hand, on $\overline{\Omega}_5$, the law of large numbers for martingales gives that

$$\lim_{t\to\infty} \frac{M(t)}{\langle M\rangle(t)} = 0, \quad \text{a.s. on } \overline{\Omega}_5.$$

Therefore, by (4.3), we may conclude that

$$\liminf_{t\to\infty} \frac{\log|X(t)|^2}{\langle M\rangle(t)} \geq \frac{\theta}{4}, \quad \text{a.s. on } \overline{\Omega}_5.$$

Hence $\lim_{t\to\infty} |X(t)| = \infty$, a.s. on $\overline{\Omega}_5$.

Since $\overline{\Omega}_3 = \Omega_5 \cup \overline{\Omega}_5$, we have that $\liminf_{t\to\infty} |X(t)| > 0$, a.s. on $\overline{\Omega}_3$, completing the proof.

5. Non–destabilisation of scalar equation

In the last section, we saw that genuinely *finite–dimensional* equations can be destabilised by a Brownian noise which preserves equilibrium stability. In this section, we show that such an equilibrium–preserving destabilisation is impossible for *scalar* equations, if noise perturbations are restricted to the class $g(X(t))\,dB(t)$.

Consider a scalar deterministic differential equation

$$\dot{x}(t) = f(x(t)), \quad t > 0; \quad x(0) = \xi \tag{5.1}$$

which has a globally asymptotically stable equilibrium at 0. We require that solutions be unique. Suitable hypotheses on f which ensure this are

$$f(0) = 0, \quad f \text{ is locally Lipschitz continuous.} \tag{5.2}$$

We now claim that the hypothesis of global stability forces f to obey

$$xf(x) < 0, \quad x \neq 0. \tag{5.3}$$

Lemma 5.1. *Suppose that the zero solution of (5.1) is globally asymptotically stable, and that f obeys (5.2). Then f must also obey (5.3).*

The proof is easy and given in the Appendix.

Suppose now that $g : \mathbb{R} \to \mathbb{R}^{1\times r}$ is any function such that

$$g(0) = 0, \quad g \text{ is locally Lipschitz continuous,} \tag{5.4}$$

and consider the stochastic differential equation

$$dX(t) = f(X(t))\,dt + g(X(t))\,dB(t) \tag{5.5}$$

with $X(0) = \xi \neq 0$.

With a perturbation of this type, solutions of the stochastic equation are also (almost surely) globally asymptotically stable.

Theorem 5.2. *Suppose that the zero equilibrium of (5.1) is globally asymptotically stable, and that f obeys (5.2). Then, for any function $g : \mathbb{R} \to \mathbb{R}$ which obeys (5.4), there is a unique continuous adapted process X which is a global solution of (5.5) and moreover obeys*

$$\lim_{t\to\infty} X(t,\xi) = 0, \quad a.s.$$

Proof. By the local Lipschitz restrictions on f and g, it follows from Proposition 2.4 that $\vartheta_0^\xi > \tau_e^\xi$, a.s. Moreover, because $|g(x)|^2 \geq 0$, and f obeys (5.3), the condition (3.1) in Assumption 3.1 is satisfied. Therefore, by Proposition 3.3, X is a global solution of (5.5), i.e., $\tau_e^\xi = \infty$, a.s. Let us assume, without loss of generality, that $\xi > 0$. Then $X(t,\xi) > 0$ for all $t \geq 0$, a.s. Scrutiny of the proof of Theorem 3.4 (without requiring Assumption 3.2) reveals that we must have

$$\lim_{t\to\infty} X(t,\xi) \in [0,\infty) \quad \text{exists a.s. and is a.s. finite.}$$

Now, remember from the proof of Theorem 3.4 that

$$X(t)^\theta = \xi^\theta + \int_0^t p(X(s))\,ds + M(t) \tag{5.6}$$

where

$$p(x) = \frac{\theta}{2}x^{\theta-2}[2xf(x) - (1-\theta)|g(x)|^2] \leq 0$$

and M is the local martingale

$$M(t) = \int_0^t \theta X(s)^{\theta-1} g(X(s))\,dB(s).$$

By Lemma 2.1, $\lim_{t\to\infty}\int_0^t p(s)ds$ is finite a.s. Suppose now that $\mathbb{P}[\Omega_1] > 0$, where $\Omega_1 = \{\omega\in\Omega : \lim_{t\to\infty} X(t,\omega) = \hat{x}(\omega) > 0\}$. By the continuity of p, for all $\omega \in \Omega_1$

$$\lim_{t\to\infty} p(X(t,\omega)) = p(\hat{x}(\omega)).$$

Notice that because $\hat{x}(\omega) > 0$, (5.3) implies that $p(\hat{x}(\omega)) < 0$. Therefore, for $\omega \in \Omega_1$,

$$\lim_{t\to\infty}\int_0^t p(X(s,\omega))ds = -\infty$$

which is a contradiction. $\square$

Appendix

Proof of Proposition 2.4

Let $A_\xi = \{\omega : X(t,\xi)(\omega) = 0 \quad \text{for some } t \in [0, \tau_e^\xi(\omega))\}$, and suppose that there exists $\zeta \in \mathbb{R}^d$ such that $\mathbb{P}[A_\zeta] > 0$. Then, there exists $T > 0$ such that $\mathbb{P}[A_\zeta(T)] > 0$, where

$$A_\zeta(T) = \{\omega : X(t,\zeta)(\omega) = 0 \quad \text{for some } t \in [0, \tau_e^\zeta(\omega) \wedge T]\}.$$

Next, let $\theta > 1$ and define

$$C_\theta = \{\omega : |X(t,\zeta)(\omega)| \le \theta - 1 \quad \text{for all } t \in [0, \tau_e^\zeta(\omega) \wedge T]\} \cap A_\zeta(T).$$

Noting that for every $\omega \in A_\zeta(T)$, the sample path $X(t,\zeta)(\omega)$ for $t \in [0, \tau_e^\zeta(\omega) \wedge T]$ is bounded, we observe that $\lim_{\theta\to\infty} C_\theta = A_\zeta(T)$. Hence, we can find a number $\bar\theta > 1$ such that $\mathbb{P}[C_\theta] > 0$ for all $\theta > \bar\theta$. Next, we introduce $\varepsilon > 0$ and the stopping time

$$\tau_\varepsilon = \inf\{t \in [0, \tau_e^\zeta) : |X(t,\zeta)| \le \varepsilon \quad \text{or} \quad |X(t,\zeta)| \ge \theta\}.$$

We consider the case for general $r \ge 1$. By Itô's rule, we get

$$\begin{aligned} |X(T\wedge\tau_\varepsilon)|^2 = |\zeta|^2 &+ \int_0^{T\wedge\tau_\varepsilon} \left(2\langle X(s), f(X(s))\rangle + |g(X(s))|_F^2\right) ds \\ &+ \sum_{j=1}^r \int_0^{T\wedge\tau_\varepsilon} \sum_{i=1}^d 2X_i(s) g_{ij}(X(s))\, dB_j(s), \end{aligned}$$

where we use the convention that $X_i(t) = \langle X(t), \mathbf{e}_i\rangle$. For $x > 0$, define $V_0(x) = x^{-1/2}$. Then applying Itô's rule in the scalar case gives

$$\begin{aligned} V_0(|X(T\wedge\tau_\varepsilon)|^2) &= V_0(|\zeta|^2) + \int_0^{T\wedge\tau_\varepsilon} V_0'(|X(s)|^2)\, d|X(s)|^2 \\ &+ \frac{1}{2}\int_0^{T\wedge\tau_\varepsilon} V_0''(|X(s)|^2) \sum_{j=1}^r \left(\sum_{i=1}^d 2X_i(s) g_{ij}(X(s))\right)^2 ds. \end{aligned} \tag{A.1}$$

For $x \in \mathbb{R}^d \setminus \{0\}$, define the functions $V(x) = 1/|x|$,

$$m(x) = -\frac{\langle x, f(x)\rangle}{|x|^2} - \frac{|g(x)|_F^2}{2|x|^2} + \frac{3}{8|x|^4} \sum_{j=1}^r \left(\sum_{i=1}^d 2x_i g_{ij}(x)\right)^2,$$

and

$$m_j(x) = -\frac{1}{|x|^2} \sum_{j=1}^r \sum_{i=1}^d x_i g_{ij}(x)$$

for $j = 1, \ldots, r$. Here we have adopted the usual notational convention $x_i = \langle x, \mathbf{e}_i \rangle$. With these functions defined, we may write (A.1) as

$$V(X(T \wedge \tau_\varepsilon)) = V(\zeta) + \int_0^{T \wedge \tau_\varepsilon} V(X(s)) m(X(s))\, ds + \sum_{j=1}^{r} \int_0^{T \wedge \tau_\varepsilon} V(X(s)) m_j(X(s))\, dB_j(s).$$

Since $f(0) = 0$, $g(0) = 0$ and f and g are locally Lipschitz continuous, it is easy to see that

$$\max_{|x| \leq \theta} m(x) =: \mu_\theta < \infty.$$

Hence, using this inequality and the optional sampling theorem (see e.g. [16, 12]), we get

$$\mathbb{E}[V(X(T \wedge \tau_\varepsilon))] \leq V(\zeta) + \mu_\theta \mathbb{E} \int_0^{T \wedge \tau_\varepsilon} V(X(s))\, ds.$$

Since V is positive, we have

$$\int_0^{\tau_\varepsilon \wedge T} V(X(s))\, ds \leq \int_0^T V(X(s \wedge \tau_\varepsilon))\, ds,$$

and so we arrive at

$$\mathbb{E}[V(X(T \wedge \tau_\varepsilon))] \leq V(\zeta) + \mu_\theta \int_0^T \mathbb{E}[V(X(s \wedge \tau_\varepsilon))]\, ds.$$

By Gronwall's inequality,

$$\mathbb{E}\left[\frac{1}{|X(T \wedge \tau_\varepsilon)|}\right] \leq \frac{1}{|\zeta|} e^{\mu_\theta T}.$$

Therefore, because $|X(T \wedge \tau_\varepsilon)| > 0$, we have

$$\mathbb{E}\left[\frac{1}{|X(T \wedge \tau_\varepsilon)|} 1_{C_\theta}\right] \leq \mathbb{E}\left[\frac{1}{|X(T \wedge \tau_\varepsilon)|}\right] \leq \frac{1}{|\zeta|} e^{\mu_\theta T}.$$

Now, let $\omega \in C_\theta$. Then, as $|X(t)| \leq \theta - 1$ for $t \in [0, \tau_e^\zeta)$, $X(\tau_\varepsilon) = \varepsilon$. If $T \wedge \tau_\varepsilon = T$, then $\tau_\varepsilon \geq T$. But the definition of C_θ means that $X(t) = 0$ for some $t < T$. Therefore, by continuity of $t \mapsto X(t)$, we have that $\tau_\varepsilon \leq T$. Hence on C_θ, $X(T \wedge \tau_\varepsilon) = \varepsilon$. Thus

$$\mathbb{E}\left[\frac{1}{|X(T \wedge \tau_\varepsilon)|} 1_{C_\theta}\right] \geq \mathbb{P}[C_\theta] \frac{1}{\varepsilon}.$$

Thus $\mathbb{P}[C_\theta] \leq \varepsilon |\zeta|^{-1} e^{\mu_\theta T}$. Letting $\varepsilon \to 0^+$ gives $\mathbb{P}[C_\theta] = 0$, a contradiction. Thus $\mathbb{P}[A_\xi] = 0$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$.

Proof of Proposition 3.3

We start by defining the sequence of stopping times $\tau_k^\xi = \inf\{t \in [0, \tau_e^\xi) : |X(t, \xi)| = k\}$. Thus, as τ_k^ξ is increasing, we may define $\tau_\infty^\xi = \lim_{k \to \infty} \tau_k^\xi$. Clearly, $\tau_\infty^\xi = \tau_e^\xi$. Suppose that $\tau_\infty^\xi = \infty$ a.s. is false. That is $\tau_\infty^\zeta < \infty$ for some $\zeta \in \mathbb{R}^d$ with positive probability. Thus, there is $T > 0$, $\varepsilon \in (0,1)$ such that

$$\mathbb{P}[\tau_k^\zeta \le T] \ge \varepsilon, \quad \forall k \ge k_0$$

for $k_0 \in \mathbb{N}$ sufficiently large. Let $C_k = \{\tau_k^\zeta \le T\}$.

Because $|X(t)| > 0$ for all $t \in [0, \tau_k^\zeta]$, we may apply Itô's rule to get

$$|X(T\wedge\tau_k^\zeta)|^2 = |\zeta|^2 + \int_0^{T\wedge\tau_k^\zeta} \left(2\langle X(s), f(X(s))\rangle + |g(X(s))|_F^2\right) ds \\ + \int_0^{T\wedge\tau_k^\zeta} 2\langle X(s), g(X(s))dB(s)\rangle.$$

Let $\theta \in (0,1)$ be the number in (3.1). Then, we may apply Itô's rule to the positive process $|X|^2$, using the fact that $|x|^\theta = (|x|^2)^{\theta/2}$, to get

$$|X(T\wedge\tau_k^\zeta)|^\theta = |\zeta|^\theta + \int_0^{T\wedge\tau_k^\zeta} \frac{\theta}{2}|X(s)|^{\theta-4}q(X(s))\, ds \\ + \int_0^{T\wedge\tau_k^\zeta} \theta|X(s)|^{\theta-2}\langle X(s), g(X(s))dB(s)\rangle,$$

where q is as defined in (3.4). Because $q(x) \le 0$ for $|x| \ne 0$, we have

$$|X(T\wedge\tau_k^\zeta)|^\theta \le |\zeta|^\theta + \int_0^{T\wedge\tau_k^\zeta} \theta|X(s)|^{\theta-2}\langle X(s), g(X(s))dB(s)\rangle.$$

Using the optional sampling theorem (see e.g. [16, 12]), we get $\mathbb{E}[|X(T\wedge\tau_k^\zeta)|^\theta] \le |\zeta|^\theta$. Now,

$$|\zeta|^\theta \ge \mathbb{E}[|X(T\wedge\tau_k^\zeta)|^\theta] \ge \mathbb{E}[|X(T\wedge\tau_k^\zeta)|^\theta 1_{C_k}].$$

If $\omega \in C_k$, then $\tau_k^\zeta \le T$, so $T\wedge\tau_k^\zeta = \tau_k^\zeta$. Thus $|X(T\wedge\tau_k^\zeta)|^\theta = k^\theta$, and so

$$\mathbb{E}[|X(T\wedge\tau_k^\zeta)|^\theta 1_{C_k}] = k^\theta\mathbb{E}[1_{C_k}] = k^\theta\mathbb{P}[C_k] \ge k^\theta\varepsilon.$$

Thus $|\zeta|^\theta \ge k^\theta\varepsilon$. Letting $k \to \infty$ yields a contradiction, so we must have $\tau_\infty^\xi = \infty$ a.s. for each $\xi \in \mathbb{R}^d$. Hence $\tau_e^\xi = \infty$ a.s. for each $\xi \in \mathbb{R}^d$, as required.

Proof of Lemma 5.1

Suppose to the contrary that $x \mapsto xf(x)$ is not negative on $\mathbb{R}\setminus\{0\}$. This means that there exists $x_1 \ne 0$ such that $x_1 f(x_1) \ge 0$. Without loss of generality, let $x_1 > 0$, so that $f(x_1) \ge 0$. Consider $\xi > x_1$. By the hypothesis of global asymptotic stability of 0, we must have

$$\lim_{t\to\infty} x(t,\xi) = 0.$$

However, we will now show that $\liminf_{t\to\infty} x(t,\xi) \ge x_1 > 0$, which contradicts the assumption that such an x_1 exists.

Consider first the case when $f(x_1) = 0$. Suppose now that there exists $t_1 > 0$ such that $x(t_1) = x_1$. (If this is not true, then $x(t,\xi) > x_1$ for all $t \geq 0$, and $\liminf_{t\to\infty} x(t,\xi) \geq x_1$ automatically.) Then, by uniqueness of solutions, $x(t) = x_1$ for all $t \geq t_1$, and so $\liminf_{t\to\infty} x(t,\xi) = x_1$.

Consider now the case when $f(x_1) > 0$. Suppose that there exists $t_0 > 0$ such that $x(t_0,\xi) < x_1$. Then there must exist a $t_1 < t_0$ such that $x(t_1,\xi) = x_1$, and $\dot{x}(t_1,\xi) \leq 0$. However, $\dot{x}(t_1) = f(x_1) > 0$, a contradiction.

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